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MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2024 年試題及參考答案
2024 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

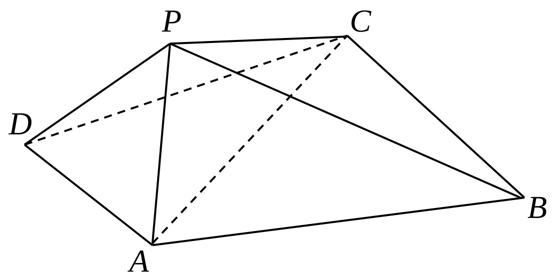
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖，$ABCD$ 是四邊形，ABC 是等邊三角形，PAD 是以 PD 為斜邊的等腰直角三角形。$\angle PAC = \frac{\pi}{2}$，$\angle PCA = \frac{\pi}{6}$，$|PA| = 1$ 及 $|CD| = 2$。

(a) 證明 DA 與 PC 垂直。 (6 分)

(b) 求三棱錐 $PBCD$ 的體積。 (6 分)

(c) 求平面 APD 和平面 CPD 的二面角的餘弦值。

[提示: 設 M 為 DP 的中點。] (8 分)

In the above figure, $ABCD$ is a quadrilateral, ABC is an equilateral triangle, PAD is a right-angled isosceles triangle with hypotenuse PD. $\angle PAC = \frac{\pi}{2}$, $\angle PCA = \frac{\pi}{6}$, $|PA| = 1$ and $|CD| = 2$.

(a) Show that DA and PC are perpendicular. (6 marks)

(b) Find the volume of the triangular pyramid $PBCD$. (6 marks)

(c) Find the cosine of the dihedral angle between plane APD and plane CPD.

[Hint. Let M be the mid-point of DP.] (8 marks)

2. (a) 一正圓錐的斜高為 1 米。設其底半徑為 x 米及體積為 $V(x)$ 立方米。

(i) 證明 $V^2(x) = \frac{\pi^2}{9}(x^4 - x^6)$，$0 \le x \le 1$。 (2 分)

(ii) 求當 $0 < x < 1$ 時 $V^2(x)$ 的局部極大值和局部極小值。 (4 分)

(iii) 求曲線 $y = V^2(x)$ 的拐點。 (3 分)

(iv) 運用 (ii) 和 (iii) 的結果，繪出曲線 $y = V^2(x)$。 (2 分)

(v) 正圓錐的最大體積是多少? (1 分)

(b) 設 k 是正常數。若在第一象限中，由直線 $y = x$ 及兩條曲線 $y = \frac{x^2}{k}$ 和 $y = \frac{x^2}{2k}$ 所包圍的區域的面積是 1，求 k 的值。 (8 分)

(a) The slant height of a right circular cone is 1 m. Suppose its base radius is x m and its volume is $V(x)$ m^3.

(i) Show that $V^2(x) = \frac{\pi^2}{9}(x^4 - x^6)$, $0 \le x \le 1$. (2 marks)

(ii) Find the local maximum and local minimum values of $V^2(x)$ when $0 < x < 1$. (4 marks)

(iii) Find the inflection point(s) of the curve $y = V^2(x)$. (3 marks)

(iv) Using the results in (ii) – (iii), sketch the curve $y = V^2(x)$. (2 marks)

(v) What is the maximum possible volume of the cone? (1 mark)

(b) Let k be a positive constant. Suppose, in the first quadrant, the area of the region bounded by the line $y = x$ and the two curves $y = \frac{x^2}{k}$ and $y = \frac{x^2}{2k}$ is 1. Find the value of k. (8 marks)

3. 已知雙曲線 $H: x^2 - \frac{y^2}{4} = 1$。有過點 $P(\sqrt{5}, 0)$ 的非垂直直線 L 與 H 交於不同的兩點 $A(x_1, y_1)$ 和 $B(x_2, y_2)$。設 m 為 L 的斜率。

(a) 證明 x_1 和 x_2 滿足方程 $(m^2 - 4)x^2 - 2\sqrt{5}m^2x + (5m^2 + 4) = 0$ 。 (2 分)

(b) 求 m 的取值範圍。 (4 分)

(c) 設 O 為原點。求 m 的值使得 $OA \perp OB$。 (6 分)

(d) 若 $m = \sqrt{5}$，求三角形 AOB 的面積。

[提示: 線段 OP 把三角形 AOB 分成兩個三角形。] (8 分)

Given a hyperbola $H: x^2 - \frac{y^2}{4} = 1$. A non-vertical line L passing through the point $P(\sqrt{5}, 0)$ interests with H at two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$. Let m be the slope of L.

(a) Show that x_1 and x_2 satisfy the equation

$$(m^2 - 4)x^2 - 2\sqrt{5}m^2x + (5m^2 + 4) = 0.$$ (2 marks)

(b) Find the range of m. (4 marks)

(c) Let O be the origin. Find the value(s) of m such that $OA \perp OB$. (6 marks)

(d) Suppose $m = \sqrt{5}$. Find the area of the triangle AOB.

[Hint. The segment OP divides the triangle AOB into two triangles.] (8 marks)

4. 設 $i=\sqrt{-1}$。

(a) (i) 以極式 $r(cos\,\theta + i\,sin\,\theta)$ 表 $\frac{\sqrt{3}+i}{1+i}$，其中 $r \geq 0$ 及 $-\pi < \theta \leq \pi$。 (4 分)

(ii) 求 $\left(\frac{\sqrt{3}+i}{1+i}\right)^{26}$，答案以 $a+bi$ 形式表示，其中 a 和 b 為實數。 (4 分)

(b) 設 $\omega = \cos\alpha + i\sin\alpha$。用棣美弗定理，證明對任意正整數 n，

$$\cos n\alpha = \frac{1}{2}(\omega^n + \omega^{-n}) \quad 及 \quad \sin n\alpha = \frac{1}{2i}(\omega^n - \omega^{-n})$$ 。

推導出 $\sin^2\alpha\cos^3\alpha = \frac{1}{16}(2\cos\alpha - \cos 3\alpha - \cos 5\alpha)$。 (8 分)

(c) 求方程 $2\cos\alpha - \cos 3\alpha - \cos 5\alpha = 0$ 的通解。 (4 分)

Let $i=\sqrt{-1}$.

(a) (i) Express $\frac{\sqrt{3}+i}{1+i}$ in polar form $r(cos\,\theta + i\,sin\,\theta)$, where $r \geq 0$ and $-\pi < \theta \leq \pi$. (4 marks)

(ii) Find $\left(\frac{\sqrt{3}+i}{1+i}\right)^{26}$. Express your answer in the form $a+bi$, where a and b are real numbers. (4 marks)

(b) Let $\omega = \cos\alpha + i\sin\alpha$. Using De Moivre's theorem, show that for any positive integer n,

$$\cos n\alpha = \frac{1}{2}(\omega^n + \omega^{-n}) \quad \text{and} \quad \sin n\alpha = \frac{1}{2i}(\omega^n - \omega^{-n}).$$

Deduce that $\sin^2\alpha\cos^3\alpha = \frac{1}{16}(2\cos\alpha - \cos 3\alpha - \cos 5\alpha)$. (8 marks)

(c) Find the general solution of the equation $2\cos\alpha - \cos 3\alpha - \cos 5\alpha = 0$. (4 marks)

5. (a) 因式分解行列式 $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ 1+b & 1+c & 1+a \end{vmatrix}$。 (7 分)

(b) 設 k 為常數。已知以 x、y 和 z 為未知量的方程組:

$$(E): \begin{cases} x+y+z=1 \\ kx+5y+z=3 \\ x+ky-z=1 \end{cases}$$ 。

(i) 求 k 的取值範圍，使得 (E) 有唯一解。 (5 分)

(ii) 設 $k=3$，求 (E) 的通解。 (4 分)

(c) 求 a 的最大值使得方程組

$$\begin{cases} x+y+z=1 \\ 3x+5y+z=3 \\ x+3y-z=1 \\ 2x+2\sqrt{y}+3z=a \end{cases}$$

有解，並當 a 取此值時，解此方程組。 (4 分)

(a) Factorize the determinant $\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ 1+b & 1+c & 1+a \end{vmatrix}$. (7 marks)

(b) Let k be a constant. Given the system of equations with unknowns x, y and z:

$$(E): \begin{cases} x+y+z=1 \\ kx+5y+z=3 \\ x+ky-z=1 \end{cases}.$$

(i) Find the range of k such that (E) has a unique solution. (5 marks)

(ii) Suppose $k=3$. Find the general solution of (E). (4 marks)

(c) Find the maximum value of a such that the system of equations

$$\begin{cases} x+y+z=1 \\ 3x+5y+z=3 \\ x+3y-z=1 \\ 2x+2\sqrt{y}+3z=a \end{cases}$$

has a solution. For this value of a, solve the system of equations. (4 marks)

參考答案：

1. (a) 從

$$|AC|=\frac{|AP|}{\tan(\angle PCA)}=\frac{1}{\tan\frac{\pi}{6}}=\sqrt{3}\text{，} \qquad (1)$$

得 $|AD|^2+|AC|^2=4=|CD|^2$，故

$$\angle DAC=\frac{\pi}{2}\text{。} \qquad (2)$$

由 $DA\perp AC$ 和 $DA\perp AP$，得 $DA\perp$ 平面 APC，故 $DA\perp PC$。

(b) 由 (2) 得 $\sin(\angle DCA)=\frac{|AD|}{|DC|}=\frac{1}{2}$，因此 $\angle DCA=\frac{\pi}{6}$。

因 ΔABC 是等邊三角形，有 $\angle ACB=\frac{\pi}{3}$。故 $\angle DCB=\angle DCA+\angle ACB=\frac{\pi}{2}$ 。

用 (1)，因 $|BC|=|AC|=\sqrt{3}$，ΔDCB 的面積是 $\frac{1}{2}|DC||BC|=\sqrt{3}$。

$PBCD$ 的體積是 $\frac{1}{3}|PA|(\Delta DCB$ 的面積$)=\frac{\sqrt{3}}{3}$。

(c) 由 M 為 DP 的中點 和 $|DA|=|AP|$，得 $AM\perp DP$。

由 (1) 得

$$|CP|=\sqrt{|AP|^2+|AC|^2}=2=|CD|\text{。} \qquad (3)$$

連同 M 為 DP 的中點，得 $CM\perp DP$。

故 $\angle CMA$ 為所求二面角的平面角。

M 為 DP 的中點，得 $|PM|=\frac{1}{2}|DP|=\frac{1}{2}\sqrt{|AD|^2+|AP|^2}=\frac{\sqrt{2}}{2}$。

由 $\angle PMA$ 是直角，得$|MA|=\sqrt{|AP|^2-|PM|^2}=\frac{\sqrt{2}}{2}$。

由 $\angle PMC$ 是直角及 (3)，得 $|MC|=\sqrt{|CP|^2-|PM|^2}=\frac{\sqrt{14}}{2}$。

故 $\cos(\angle CMA)=\frac{|MA|^2+|MC|^2-|AC|^2}{2|MA||MC|}=\frac{\sqrt{7}}{7}$。

[註: 可證明 $\angle MAC=\frac{\pi}{2}$，然後 $\cos(\angle CMA)=\frac{|MA|}{|MC|}$。]

2. (a)(i) 設正圓錐的高為 h 米。由 $\begin{cases}V=\frac{1}{3}\pi x^2h\\x^2+h^2=1\end{cases}$，得 $V^2(x)=\frac{\pi^2}{9}x^4(1-x^2)=\frac{\pi^2}{9}(x^4-x^6)$。

(ii) $\frac{dV^2}{dx}=\frac{2\pi^2}{9}(2x^3-3x^5)$。當 $0<x<1$，$\frac{dV^2}{dx}=0\Leftrightarrow 2x^3-3x^5=0\Leftrightarrow x=\frac{\sqrt{6}}{3}$。

當 $0<x<\frac{\sqrt{6}}{3}$，$\frac{dV^2}{dx}>0$，故 $V^2(x)$ 是遞增的。

當 $\frac{\sqrt{6}}{3}<x<1$，$\frac{dV^2}{dx}<0$ ，故 $V^2(x)$ 是遞減的。

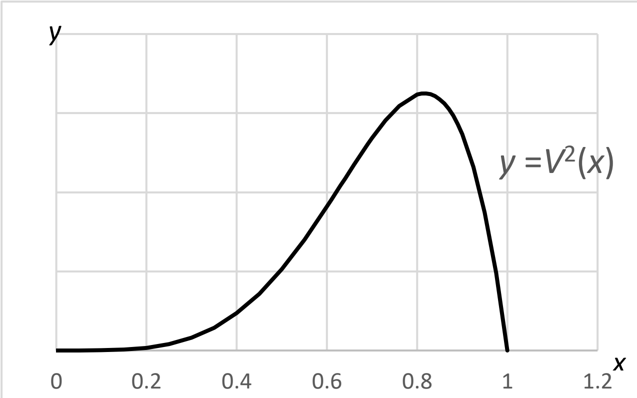
因此，$V^2\left(\frac{\sqrt{6}}{3}\right)=\frac{4\pi^2}{243}$ 是一局部極大值。

(iii) $\frac{d^2V^2}{dx^2}=\frac{2\pi^2}{3}(2x^2-5x^4)$。 當 $0<x<1$，$\frac{d^2V^2}{dx^2}=0\Leftrightarrow 2x^2-5x^4=0\Leftrightarrow x=\frac{\sqrt{10}}{5}$。

當 $0<x<\frac{\sqrt{10}}{5}$，$\frac{d^2V^2}{dx^2}>0$。當 $\frac{\sqrt{10}}{5}<x<1$，$\frac{d^2V^2}{dx^2}<0$。

因此，$\left(\frac{\sqrt{10}}{5},\frac{4\pi^2}{375}\right)$ 是曲線 $y=V^2(x)$ 的拐點。

(iv)



(v) $\frac{2\sqrt{3}\pi}{27}$ 立方米

(b) 解 $\begin{cases} y = x \\ y = \frac{x^2}{k} \end{cases}$ 得 $x = 0$ 或 $x = k$。 解 $\begin{cases} y = x \\ y = \frac{x^2}{2k} \end{cases}$ 得 $x = 0$ 或 $x = 2k$。

從所包圍的區域的面積為 1，得

$$\int_0^k \frac{x^2}{k} - \frac{x^2}{2k} dx + \int_k^{2k} x - \frac{x^2}{2k} dx = 1 \Rightarrow \left[\frac{x^3}{6k}\right]_0^k + \left[\frac{x^2}{2} - \frac{x^3}{6k}\right]_k^{2k} = 1$$
$$\Rightarrow \frac{k^2}{6} + \left[\left(2k^2 - \frac{4k^2}{3}\right) - \left(\frac{k^2}{2} - \frac{k^2}{6}\right)\right] = 1$$
$$\Rightarrow k = \sqrt{2}\text{。} \quad (k > 0)$$

3. (a) 由 $\begin{cases} x^2 - \frac{y^2}{4} = 1 \\ y = m(x - \sqrt{5}) \end{cases}$ ，得 $4x^2 - m^2(x - \sqrt{5})^2 = 4$，即

$$(m^2 - 4)x^2 - 2\sqrt{5}m^2x + 5m^2 + 4 = 0\text{。} \qquad (1)$$

(b) 因非垂直直線 L 與 H 有兩不同的交點，(1) 有兩不同的實根，故 $m^2 - 4 \neq 0$ 及 (1) 的判別式大於 0，即 $m \neq \pm 2$ 及

$$\left(-2\sqrt{5}m^2\right)^2 - 4(m^2 - 4)(5m^2 + 4) > 0\text{。} \qquad (2)$$

由 (2) 得 $64m^2 + 64 > 0$，此不等式對所有 m 都成立，故 m 的取值範圍是 $\mathbb{R} \setminus \{\pm 2\}$。

(c) 由 (1) 得

$$x_1 + x_2 = \frac{2\sqrt{5}m^2}{m^2 - 4} \text{ 及 } x_1x_2 = \frac{5m^2 + 4}{m^2 - 4}\text{。} \qquad (3)$$

因此，

$$
\begin{aligned}
OA \perp OB &\Rightarrow \frac{y_1}{x_1}\cdot\frac{y_2}{x_2} = -1 \\
&\Rightarrow \left[m\left(x_1-\sqrt{5}\right)\right]\left[m\left(x_2-\sqrt{5}\right)\right] + x_1x_2 = 0 \\
&\Rightarrow (m^2+1)(x_1x_2) - \sqrt{5}m^2(x_1+x_2) + 5m^2 = 0 \\
&\Rightarrow (m^2+1)\left(\frac{5m^2+4}{m^2-4}\right) - \sqrt{5}m^2\left(\frac{2\sqrt{5}m^2}{m^2-4}\right) + 5m^2 = 0 \\
&\Rightarrow (m^2+1)(5m^2+4) - 10m^4 + 5m^2(m^2-4) = 0 \\
&\Rightarrow 11m^2 = 4 \\
&\Rightarrow m = \pm\frac{2\sqrt{11}}{11}\text{。}
\end{aligned}
$$

(d) 當 $m=\sqrt{5}$，代入 (3) 式得 $x_1+x_2=10\sqrt{5}$ 及 $x_1x_2=29$，故 x_1 和 x_2 為正數，
從而得知點 A 和點 B 在同一分支上。不妨假設 $y_1>0$ 和 $y_2<0$，則有

$$
\begin{aligned}
\Delta AOB \text{ 的面積} &= \Delta AOP \text{ 的面積} + \Delta BOP \text{ 的面積} \\
&= \frac{1}{2}|OP|y_1 + \frac{1}{2}|OP|(-y_2) = \frac{1}{2}\sqrt{5}(y_1-y_2)\text{。}
\end{aligned}
$$

計算得

$$
\begin{aligned}
(y_1-y_2)^2 &= (y_1+y_2)^2 - 4y_1y_2 \\
&= \left[\sqrt{5}\left(x_1-\sqrt{5}\right)+\sqrt{5}\left(x_2-\sqrt{5}\right)\right]^2 - 4\left[\sqrt{5}\left(x_1-\sqrt{5}\right)\right]\left[\sqrt{5}\left(x_2-\sqrt{5}\right)\right] \\
&= \left[\sqrt{5}(x_1+x_2)-10\right]^2 - 20\left[x_1x_2-\sqrt{5}(x_1+x_2)+5\right] \\
&= 1920\text{。}
\end{aligned}
$$

ΔAOB 的面積是 $20\sqrt{6}$。

4. (a) (i) $\frac{\sqrt{3}+i}{1+i} = \frac{2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})}{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})} = \sqrt{2}[\cos(\frac{\pi}{6}-\frac{\pi}{4}) + i\sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)] = \sqrt{2}[\cos(-\frac{\pi}{12}) + i\sin\left(-\frac{\pi}{12}\right)]$

(ii)

$$
\begin{aligned}
\left(\frac{\sqrt{3}+i}{1+i}\right)^{26} &= \left(\sqrt{2}\right)^{26}[\cos(-\tfrac{\pi}{12}) + i\sin\left(-\tfrac{\pi}{12}\right)]^{26} \\
&= 2^{13}[\cos\left(-\tfrac{26\pi}{12}\right) + i\sin\left(-\tfrac{26\pi}{12}\right)] \\
&= 2^{13}[\cos(-\tfrac{\pi}{6}) + i\sin\left(-\tfrac{\pi}{6}\right)] \\
&= 2^{12}\sqrt{3} - 2^{12}i\text{。}
\end{aligned}
$$

(b) 由

$$\omega^n = \cos n\alpha + i\sin n\alpha \quad \text{及} \quad \omega^{-n} = \cos(-n\alpha) + i\sin(-n\alpha) = \cos n\alpha - i\sin n\alpha$$

的和與差可得出結果。

$$
\begin{aligned}
\sin^2\alpha\cos^3\alpha &= \left[\frac{1}{2i}(\omega-\omega^{-1})\right]^2\left[\frac{1}{2}(\omega+\omega^{-1})\right]^3 \\
&= -\frac{1}{32}(\omega^2-2+\omega^{-2})(\omega^3+3\omega+3\omega^{-1}+\omega^{-3}) \\
&= -\frac{1}{32}[(\omega^5+\omega^{-5}) + (\omega^3+\omega^{-3}) - 2(\omega-\omega^{-1})] \\
&= \frac{1}{16}(2\cos\alpha - \cos 3\alpha - \cos 5\alpha)
\end{aligned}
$$

(c)

$$\begin{aligned}2\cos\alpha-\cos 3\alpha-\cos 5\alpha=0 &\Rightarrow 16\sin^2\alpha\cos^3\alpha=0\\ &\Rightarrow \sin\alpha=0 \text{ 或 } \cos\alpha=0\\ &\Rightarrow \alpha=k\pi \text{ 或 } \alpha=\frac{\pi}{2}+k\pi\text{，}k\text{ 為整數}\\ &\Rightarrow \alpha=\frac{n\pi}{2}\text{，}n\text{ 為整數。}\end{aligned}$$

5. (a)

$$\begin{aligned}\begin{vmatrix}a & b & c\\ b+c & c+a & a+b\\ 1+b & 1+c & 1+a\end{vmatrix} &= \begin{vmatrix}a & b & c\\ a+b+c & a+b+c & a+b+c\\ 1+b & 1+c & 1+a\end{vmatrix}\\ &=(a+b+c)\begin{vmatrix}a & b & c\\ 1 & 1 & 1\\ 1+b & 1+c & 1+a\end{vmatrix}\\ &=(a+b+c)\begin{vmatrix}a & b-a & c-a\\ 1 & 0 & 0\\ 1+b & c-b & a-b\end{vmatrix}\\ &=(a+b+c)\{(-1)[(b-a)(a-b)-(c-b)(c-a)]\}\\ &=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\end{aligned}$$

(b)(i) (E) 有唯一解當且僅當 $\begin{vmatrix}1 & 1 & 1\\ k & 5 & 1\\ 1 & k & -1\end{vmatrix}\neq 0$，即 $k\neq\pm 3$。

(ii) 當 $k=3$，方程組變成 $\begin{cases}x+y+z=1\\ 3x+5y+z=3\\ x+3y-z=1\end{cases}$。

解 $\begin{cases}x+y+z=1\\ 3x+5y+z=3\end{cases}$，得 $x=1-2t$，$y=t$，$z=t$，$t\in\mathbb{R}$。

此解亦滿足第三條方程，因此是方程組的通解。

(c) 設方程組有解，則有 $t\geq 0$ 使得 $2(1-2t)+2\sqrt{t}+3t=a$，即 $3-\left(1-\sqrt{t}\right)^2=a$。

由此得知 a 的最大值是 3。當 $a=3$，得 $t=1$ 及方程組的解是 $x=-1$，$y=1$，$z=1$。

Suggested Answers:

1. (a) It follows from

$$|AC| = \frac{|AP|}{\tan(\angle PCA)} = \frac{1}{\tan\frac{\pi}{6}} = \sqrt{3} \quad (1)$$

that $|AD|^2 + |AC|^2 = 4 = |CD|^2$. Hence,

$$\angle DAC = \frac{\pi}{2}. \quad (2)$$

From $DA \perp AC$ and $DA \perp AP$, we get $DA \perp$ plane APC. Hence $DA \perp PC$.

(b) From (2), we have $\sin(\angle DCA) = \frac{|AD|}{|DC|} = \frac{1}{2}$. Hence, $\angle DCA = \frac{\pi}{6}$.

Since ΔABC is equilateral, $\angle ACB = \frac{\pi}{3}$. Hence, $\angle DCB = \angle DCA + \angle ACB = \frac{\pi}{2}$.

Using (1), as $|BC| = |AC| = \sqrt{3}$, the area of ΔDCB is $\frac{1}{2}|DC||BC| = \sqrt{3}$.

The volume of $PBCD$ is $\frac{1}{3}|PA|$(area of ΔDCB) $= \frac{\sqrt{3}}{3}$.

(c) As M is the mid-point of DP and $|DA| = |AP|$, we get $AM \perp DP$.

Using (1), we get

$$|CP| = \sqrt{|AP|^2 + |AC|^2} = 2 = |CD|. \quad (3)$$

With M is the mid-point of DP, we get $CM \perp DP$. Hence, the required dihedral angle is $\angle CMA$.

As M is the mid-point of DP, we get $|PM| = \frac{1}{2}|DP| = \frac{1}{2}\sqrt{|AD|^2 + |AP|^2} = \frac{\sqrt{2}}{2}$.

As $\angle PMA$ is a right angle, we get $|MA| = \sqrt{|AP|^2 - |PM|^2} = \frac{\sqrt{2}}{2}$.

As $\angle PMC$ is a right angle, using (3), we get $|MC| = \sqrt{|CP|^2 - |PM|^2} = \frac{\sqrt{14}}{2}$.

Hence, $\cos(\angle CMA) = \frac{|MA|^2 + |MC|^2 - |AC|^2}{2|MA||MC|} = \frac{\sqrt{7}}{7}$.

[Remark: One may prove $\angle MAC = \frac{\pi}{2}$. Then $\cos(\angle CMA) = \frac{|MA|}{|MC|}$.]

2. (a)(i) Suppose the height of the right circular cone is h m.

From $\begin{cases} V = \frac{1}{3}\pi x^2 h \\ x^2 + h^2 = 1 \end{cases}$, we get $V^2(x) = \frac{\pi^2}{9}x^4(1 - x^2) = \frac{\pi^2}{9}(x^4 - x^6)$.

(ii) $\frac{dV^2}{dx} = \frac{2\pi^2}{9}(2x^3 - 3x^5)$. When $0 < x < 1$, $\frac{dV^2}{dx} = 0 \Leftrightarrow 2x^3 - 3x^5 = 0 \Leftrightarrow x = \frac{\sqrt{6}}{3}$.

When $0 < x < \frac{\sqrt{6}}{3}$, $\frac{dV^2}{dx} > 0$. So, $V^2(x)$ is increasing.

When $\frac{\sqrt{6}}{3} < x < 1$, $\frac{dV^2}{dx} < 0$. So, $V^2(x)$ is decreasing.

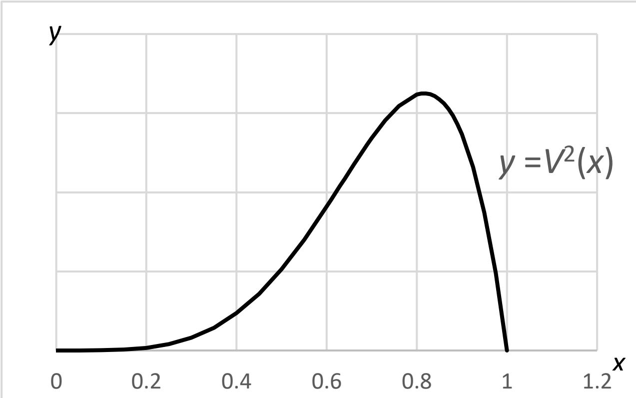
Hence, $V^2(\frac{\sqrt{6}}{3}) = \frac{4\pi^2}{243}$ is a local maximum value.

(iii) $\frac{d^2V^2}{dx^2} = \frac{2\pi^2}{3}(2x^2 - 5x^4)$. When $0 < x < 1$, $\frac{d^2V^2}{dx^2} = 0 \Leftrightarrow 2x^2 - 5x^4 = 0 \Leftrightarrow x = \frac{\sqrt{10}}{5}$.

When $0 < x < \frac{\sqrt{10}}{5}$, $\frac{d^2V^2}{dx^2} > 0$. When $\frac{\sqrt{10}}{5} < x < 1$, $\frac{d^2V^2}{dx^2} < 0$.

Hence, $\left(\frac{\sqrt{10}}{5}, \frac{4\pi^2}{375}\right)$ is an inflection point of the curve $y = V^2(x)$.

(iv)



(v) $\frac{2\sqrt{3}\pi}{27}$ m^3

(b) Solving $\begin{cases} y = x \\ y = \frac{x^2}{k} \end{cases}$, we get $x = 0$ or $x = k$. Solving $\begin{cases} y = x \\ y = \frac{x^2}{2k} \end{cases}$, we get $x = 0$ or $x = 2k$.

As the area of the bounded region is 1, we get

$$\int_0^k \frac{x^2}{k} - \frac{x^2}{2k} dx + \int_k^{2k} x - \frac{x^2}{2k} dx = 1 \Rightarrow \left[\frac{x^3}{6k}\right]_0^k + \left[\frac{x^2}{2} - \frac{x^3}{6k}\right]_k^{2k} = 1$$
$$\Rightarrow \frac{k^2}{6} + \left[\left(2k^2 - \frac{4k^2}{3}\right) - \left(\frac{k^2}{2} - \frac{k^2}{6}\right)\right] = 1$$
$$\Rightarrow k = \sqrt{2}. \quad (k > 0)$$

3. (a) From $\begin{cases} x^2 - \frac{y^2}{4} = 1 \\ y = m(x - \sqrt{5}) \end{cases}$, we get $4x^2 - m^2(x - \sqrt{5})^2 = 4$. That is,

$$(m^2 - 4)x^2 - 2\sqrt{5}m^2x + 5m^2 + 4 = 0. \qquad (1)$$

(b) As the non-vertical line L and H have two distinct intersection points, (1) has two distinct real roots. So, $m^2 - 4 \neq 0$ and the discriminant of (1) is bigger than 0. That is, $m \neq \pm 2$ and

$$\left(-2\sqrt{5}m^2\right)^2 - 4(m^2 - 4)(5m^2 + 4) > 0. \qquad (2)$$

From (2), we get $64m^2 + 64 > 0$, which is true for all m. Thus, the range of m is $\mathbb{R} \setminus \{\pm 2\}$.

(c) From (1), we get

$$x_1 + x_2 = \frac{2\sqrt{5}m^2}{m^2 - 4} \text{ and } x_1x_2 = \frac{5m^2 + 4}{m^2 - 4}. \qquad (3)$$

Hence,

$$\begin{aligned} OA \perp OB &\Rightarrow \frac{y_1}{x_1}\cdot\frac{y_2}{x_2} = -1 \\ &\Rightarrow \left[m\left(x_1-\sqrt{5}\right)\right]\left[m\left(x_2-\sqrt{5}\right)\right] + x_1x_2 = 0 \\ &\Rightarrow (m^2+1)(x_1x_2) - \sqrt{5}m^2(x_1+x_2) + 5m^2 = 0 \\ &\Rightarrow (m^2+1)\left(\frac{5m^2+4}{m^2-4}\right) - \sqrt{5}m^2\left(\frac{2\sqrt{5}m^2}{m^2-4}\right) + 5m^2 = 0 \\ &\Rightarrow (m^2+1)(5m^2+4) - 10m^4 + 5m^2(m^2-4) = 0 \\ &\Rightarrow 11m^2 = 4 \\ &\Rightarrow m = \pm\frac{2\sqrt{11}}{11}. \end{aligned}$$

(d) Substituting $m=\sqrt{5}$ into (3), we get $x_1+x_2 = 10\sqrt{5}$ and $x_1x_2 = 29$. So, x_1 and x_2 are positive numbers. Hence, we know that points A and B are on the same branch. We may assume $y_1 > 0$ and $y_2 < 0$. Then,

$$\begin{aligned} \text{area of } \Delta AOB &= \text{area of } \Delta AOP + \text{area of } \Delta BOP \\ &= \frac{1}{2}|OP|y_1 + \frac{1}{2}|OP|(-y_2) = \frac{1}{2}\sqrt{5}(y_1-y_2). \end{aligned}$$

By direct calculation,

$$\begin{aligned} (y_1-y_2)^2 &= (y_1+y_2)^2 - 4y_1y_2 \\ &= \left[\sqrt{5}\left(x_1-\sqrt{5}\right)+\sqrt{5}\left(x_2-\sqrt{5}\right)\right]^2 - 4\left[\sqrt{5}\left(x_1-\sqrt{5}\right)\right]\left[\sqrt{5}\left(x_2-\sqrt{5}\right)\right] = 0 \\ &= \left[\sqrt{5}(x_1+x_2)-10\right]^2 - 20\left[x_1x_2 - \sqrt{5}(x_1+x_2)+5\right] \\ &= 1920. \end{aligned}$$

The area of ΔAOB is $20\sqrt{6}$.

4. (a) (i) $\frac{\sqrt{3}+i}{1+i} = \frac{2(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6})}{\sqrt{2}(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4})} = \sqrt{2}[\cos(\frac{\pi}{6}-\frac{\pi}{4}) + i\sin\left(\frac{\pi}{6}-\frac{\pi}{4}\right)] = \sqrt{2}[\cos(-\frac{\pi}{12}) + i\sin\left(-\frac{\pi}{12}\right)]$

(ii)

$$\begin{aligned} \left(\frac{\sqrt{3}+i}{1+i}\right)^{26} &= \left(\sqrt{2}\right)^{26}[\cos(-\frac{\pi}{12}) + i\sin\left(-\frac{\pi}{12}\right)]^{26} \\ &= 2^{13}[\cos\left(-\frac{26\pi}{12}\right) + i\sin\left(-\frac{26\pi}{12}\right)] \\ &= 2^{13}[\cos(-\frac{\pi}{6}) + i\sin\left(-\frac{\pi}{6}\right)] \\ &= 2^{12}\sqrt{3} - 2^{12}i \text{ 。} \end{aligned}$$

(b) The results follow from the sum and difference of

$$\omega^n = \cos n\alpha + i\sin n\alpha \quad \text{and} \quad \omega^{-n} = \cos(-n\alpha) + i\sin(-n\alpha) = \cos n\alpha - i\sin n\alpha.$$

$$\begin{aligned} \sin^2\alpha\cos^3\alpha &= \left[\frac{1}{2i}(\omega-\omega^{-1})\right]^2\left[\frac{1}{2}(\omega+\omega^{-1})\right]^3 \\ &= -\frac{1}{32}(\omega^2-2+\omega^{-2})(\omega^3+3\omega+3\omega^{-1}+\omega^{-3}) \\ &= -\frac{1}{32}[(\omega^5+\omega^{-5}) + (\omega^3+\omega^{-3}) - 2(\omega-\omega^{-1})] \\ &= \frac{1}{16}(2\cos\alpha - \cos 3\alpha - \cos 5\alpha) \end{aligned}$$

(c)

$$\begin{aligned}2\cos\alpha-\cos 3\alpha-\cos 5\alpha=0 &\Rightarrow 16\sin^2\alpha\cos^3\alpha=0\\ &\Rightarrow \sin\alpha=0 \text{ or } \cos\alpha=0\\ &\Rightarrow \alpha=k\pi \text{ or } \alpha=\frac{\pi}{2}+k\pi, k \text{ is an integer}\\ &\Rightarrow \alpha=\frac{n\pi}{2}, n \text{ is an integer.}\end{aligned}$$

5. (a)

$$\begin{aligned}\begin{vmatrix} a & b & c \\ b+c & c+a & a+b \\ 1+b & 1+c & 1+a \end{vmatrix} &= \begin{vmatrix} a & b & c \\ a+b+c & a+b+c & a+b+c \\ 1+b & 1+c & 1+a \end{vmatrix}\\ &=(a+b+c)\begin{vmatrix} a & b & c \\ 1 & 1 & 1 \\ 1+b & 1+c & 1+a \end{vmatrix}\\ &=(a+b+c)\begin{vmatrix} a & b-a & c-a \\ 1 & 0 & 0 \\ 1+b & c-b & a-b \end{vmatrix}\\ &=(a+b+c)\{(-1)[(b-a)(a-b)-(c-b)(c-a)]\}\\ &=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)\end{aligned}$$

(b)(i) (E) has a unique solution if and only if $\begin{vmatrix} 1 & 1 & 1 \\ k & 5 & 1 \\ 1 & k & -1 \end{vmatrix} \neq 0$, that is, $k \neq \pm 3$.

(ii) When $k=3$, the system of equations becomes $\begin{cases} x+y+z=1 \\ 3x+5y+z=3 \\ x+3y-z=1 \end{cases}$.

Solving $\begin{cases} x+y+z=1 \\ 3x+5y+z=3 \end{cases}$, we get $x=1-2t, y=t, z=t$, $t\in\mathbb{R}$.

This solution also satisfies the third equation and hence is the general solution of the system of equations.

(c) Suppose the system of equations has a solution. Then there exists $t \geq 0$ such that $2(1-2t)+2\sqrt{t}+3t=a$, that is, $3-\left(1-\sqrt{t}\right)^2=a$. Hence, we know that the maximum value of a is 3. When $a=3$, we have $t=1$ and the solution of the system of equations is $x=-1, y=1, z=1$.