澳門四高校聯合入學考試 (語言科及數學科)

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2020 年試題及參考答案
2020 Examination Paper and Suggested Answer

數學正卷 Mathematics Standard Paper

<u>第一部份</u> **選擇題。請選出每題之*最佳答案*。**

1. 若 $P=\{0, 1, 2\}$，$Q=\{0, 2, 3\}$，$R=\{1, 2, 3, 4\}$，則 $P\cap(Q\cup R)=$

A. $\{0, 1, 2\}$　B. $\{0, 1, 2, 3, 4\}$　C. $\{2\}$
D. $\varnothing$　E. 以上皆非

2. 一個城市於 2017 年終的人口為 200 萬，若此城市的人口增長率為每年 2%，於 2020 年終的人口應約為（選擇最接近的答案）：

A. 2,100,000　B. 2,080,000　C. 2,140,000　D. 2,060,000　E. 2,120,000

3. $9x^2-3x-4y^2+2y=$

A. $(3x-2y)(3x+2y+1)$　B. $(3x-2y)(3x+2y-1)$　C. $(3x+2y)(3x-2y+1)$
D. $(2x+3y)(2x-3y-1)$　E. 以上皆非

4. 若 $\sqrt[n]{x}$ 表示 x 的 n 次根，下列哪一個正確？

A. $\sqrt[4]{17}<\sqrt{4.1}<\sqrt[3]{7.2}$　B. $\sqrt[3]{7.2}<\sqrt{4.1}<\sqrt[4]{17}$　C. $\sqrt[3]{7.2}<\sqrt[4]{17}<\sqrt{4.1}$
D. $\sqrt[4]{17}<\sqrt[3]{7.2}<\sqrt{4.1}$　E. 以上皆非

5. 若 $|-3x-5|\le 4$，則

A. $-\frac{1}{3}\le x\le 3$　B. $\frac{1}{3}\le x\le 3$　C. $-3\le x\le -\frac{1}{3}$
D. $-3\le x\le \frac{1}{3}$　E. $x\le -3$ 或 $x\ge -\frac{1}{3}$

6. 若 $x^5=y^{13}$，下列何者是真？

I. $\log_x y=\frac{13}{5}$　II. $\log_y x=\frac{13}{5}$　III. $\log_x y=\frac{5}{13}$　IV. $\log_y x=\frac{5}{13}$

A. 只有 I　B. 只有 II　C. 只有 IV
D. 只有 II 及 III　E. 只有 I 及 IV

7. 若 $\sqrt{x+7}-\sqrt{x}=2$，則 $x=$

A. $\frac{9}{16}$　B. $\frac{3}{16}$　C. $\frac{9}{4}$　D. $\frac{4}{9}$　E. $\frac{3}{4}$

8. 從 0, 1, 3, 5, 6, 7 這六個數字任取三個不同數字排成一個三位偶數，這樣的偶數有多少個？[注意：013 只是一個兩位數，而非一個三位數。]

A. 26　B. 36　C. 42　D. 45　E. 以上皆非

9. 在 $\left(x^2+\frac{1}{x}\right)^{12}$ 的展開式中，常數項是

A. 495　B. 220　C. 66　D. 792　E. 以上皆非

10. 設 $(a, 0)$ 及 $(0, -b)$ 為一圓形直徑之兩端點，下列哪點在該圓形上？

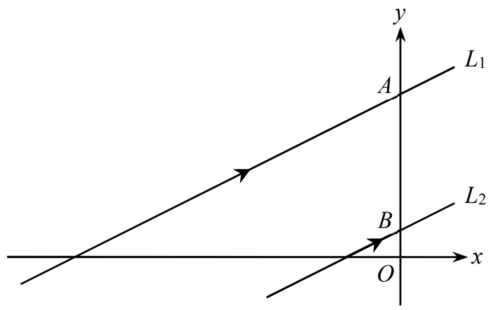
I. $(0, 0)$　II. (a, b)　III. $(a, -b)$

A. 只有 I　B. 只有 III　C. 只有 I 及 II
D. 只有 II 及 III　E. 只有 I 及 III

11. 若將拋物綫 $y=(x+1)^2$ 的圖形向右平移 4 個單位，再向下移動 5 個單位，所得的拋物綫方程為何？

A. $y=(x-3)^2-5$　B. $y=(x+5)^2-5$　C. $y=(x-5)^2+3$
D. $y=(x+3)^2-5$　E. $y=(x+4)^2-5$

12. 如右圖所示，一對平行綫 L_1: $x-4y+12=0$ 及 L_2: $2x-my+4=0$ 分別與 y 軸交於 A 和 B 兩點。求綫段 AB 的長度。

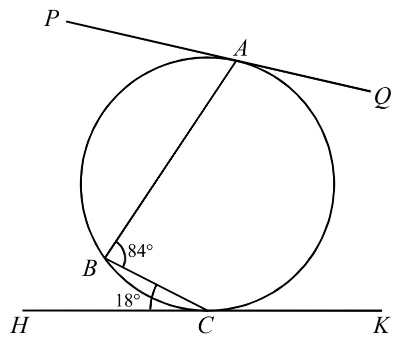


A. 4　B. 7/2
C. 5/2　D. 1
E. 以上皆非

13. 在等差數列 $\{a_n\}_{n\geq 1}$ 中，$a_2=22$ 及 $a_5=13$。此數列中有多少項為正值？

A. 6　B. 7　C. 8　D. 9　E. 10

14. 如右圖所示，PQ 和 HK 分別與圓切於點 A 和 C。已知 $\angle ABC=84°$ 及 $\angle BCH=18°$，求$\angle PAB$。



A. 45°　B. 72°
C. 78°　D. 85°
E. 以上皆非

15. 若 $\sin\theta-\cos\theta=-\frac{1}{5}$，且 $\pi<\theta<2\pi$，則 $\cos 2\theta$ 之值是多少？

A. $\frac{7}{25}$　B. $-\frac{7}{25}$　C. $\pm\frac{7}{25}$　D. $-\frac{12}{25}$　E. $\frac{12}{25}$

<u>第二部份</u> **解答題。**

1. 譚先生在他的抽屜中發現了一些含兩種牌子的電腦磁碟，其中 40% 的磁碟是牌子 A。他知道牌子 A 的磁碟全是可用的，而牌子 B 的磁碟有 40% 是可用的。

   (a) 求從這些磁碟中隨機抽出一張可用的磁碟的概率。 (4 分)

   (b) 假設這些磁碟共有 100 張。若從中隨機抽出 2 張，則它們都是可用磁碟的概率是多少？(答案以最簡分數表示。) (4 分)

2. 一列火車原本以每小時 40 公里的速度從 A 市駛向 B 市。經過 2 小時後，由於發動機故障，火車要停駛 45 分鐘，以便復修發動機。復修後它以每小時 50 公里的速度行駛，完成餘下的旅程。

   (a) 設 A 市和 B 市相距 x 公里，試用 x 表示在以上情況下火車由 A 駛到 B 所花的時間(以小時為單位)。 (5 分)

   (b) 若火車最終能準時到達目的地，從 A 市到 B 市有多遠？ (3 分)

3. 已知數列 $\{a_n\}_{n\geq 1}$ 滿足 $a_{n+1}=a_n+\dfrac{2n+1}{n^2(n+1)^2}$ 和 $a_1=0$。

   (a) 求 a_2、a_3 及 a_4 的值。 (3 分)

   (b) 猜測數列 $\{a_n\}_{n\geq 1}$ 的通項公式，並用數學歸納法證明。 (5 分)

4. 已知關於 x 的方程 $a^{2x-4}-5a^{x-2}+6=0$ 有一個根為 3，其中 a 為常數。

   (a) 求 a 的值。 (4 分)

   (b) 求關於 x 的方程其餘的根。 (4 分)

5. 設 $x^2+y^2-4x-6y-12=0$ 、 $3y=(3\tan 30°)x+9-2\sqrt{3}$ 和 $y=(\tan 60°)x-(2\sqrt{3}-3)$ 分別為圓形 C、直綫 ℓ_1 和 ℓ_2 的方程。另設 A、B 分別為 ℓ_1 和 ℓ_2 與 C 的交點，其中 A 和 B 的 x 座標均大於 3。

   (a) 求 C 的半徑 r 及圓心 P 的座標。 (2 分)

   (b) 證明 ℓ_1 和 ℓ_2 均通過 P。 (3 分)

   (c) 求 ΔPAB 的面積。 (3 分)

參考答案

<u>第一部份</u> 選擇題。

題目編號	最佳答案
1	A
2	E
3	B
4	B
5	C
6	D
7	A
8	B
9	A
10	E
11	A
12	C
13	D
14	C
15	B

(第二部份答案由下頁開始)

第二部份 **解答題。**

1. (a) 抽出一張可用磁碟的概率 $=40\%+(1-40\%)\times 40\%=0.64$。

   (b) 抽出兩張可用磁碟的概率 $=\dfrac{64}{100}\times\dfrac{63}{99}=\dfrac{112}{275}$。

2. (a) 火車在首 2 小時走了 $40\times2=80$ 公里。壞車期間，停駛了 45 分鐘。修復後，它跑了 $x-80$ 公里，這段車程花了 $\dfrac{x-80}{50}$ 小時才到達 B 市。整個車程共用了

$$2+\frac{45}{60}+\frac{x-80}{50}=\frac{23}{20}+\frac{x}{50} \text{ 小時。}$$

   (b)由於火車原本以每小時 40 公里行駛，若中途沒有耽誤，它應該用 $\dfrac{x}{40}$ 小時準時到達 B 市，所以有

$$\frac{23}{20}+\frac{x}{50}=\frac{x}{40}\ \Rightarrow x=230\text{。}$$

   即 A、B 兩市相距 230 公里。

3. (a) $a_2=0+\dfrac{2\cdot1+1}{1^2\cdot2^2}=\dfrac{3}{4}$，$a_3=\dfrac{3}{4}+\dfrac{2\cdot2+1}{2^2\cdot3^2}=\dfrac{8}{9}$，及 $a_4=\dfrac{8}{9}+\dfrac{2\cdot3+1}{3^2\cdot4^2}=\dfrac{15}{16}$。

   (b) 由 (a) 會猜測 $a_n=\dfrac{n^2-1}{n^2}$。

   設 $S(n)$ 代表命題“$a_n=\dfrac{n^2-1}{n^2}$”。

   由已知條件 $a_1=0$，可知 $S(1)$ 成立。

   假設 $S(k)$ 對某正整數 k 成立，即假定 $a_k=\dfrac{k^2-1}{k^2}$ 對這個 k 成立。

   從這個假設和另一已知條件得

$$a_{k+1}=\frac{k^2-1}{k^2}+\frac{2k+1}{k^2(k+1)^2}=\frac{(k^2-1)(k+1)^2+2k+1}{k^2(k+1)^2}=\frac{k^2+2k}{(k+1)^2}=\frac{(k+1)^2-1}{(k+1)^2}\text{。}$$

   換句話說，$S(k+1)$ 也成立。

   根據數學歸納法原理，可知 $S(n)$ 對所有正整數 n 都成立。

4. (a) 因為 $x=3$ 是方程的一個根，從而有 $a^2-5a+6=0$，即 $(a-2)(a-3)=0$，得 $a=2$ 或 $a=3$。

   (b) 當 $a=2$ 時，原方程可以化為 $2^{2(x-2)}-5\cdot2^{x-2}+6=(2^{x-2}-2)(2^{x-2}-3)=0$。解得 $2^{x-2}=2$ 或 $2^{x-2}=3$。即 $x=3$ 或 $x=\log_2 3+2$。

   當 $a=3$ 時，原方程可以化為 $3^{2(x-2)}-5\cdot3^{x-2}+6=(3^{x-2}-2)(3^{x-2}-3)=0$。解得 $3^{x-2}=2$ 或 $3^{x-2}=3$。即 $x=3$ 或 $x=\log_3 2+2$。

   所以其餘的根為$\log_2 3+2$及$\log_3 2+2$。

5. (a) 將 C 的方程寫成 $(x-2)^2+(y-3)^2=5^2$，得 $P=(2,3)$，$r=5$。

(b) 已知 $\tan 30°=1/\sqrt{3}$ 及 $\tan 60°=\sqrt{3}$。當 $x=2$ 及 $y=3$，

ℓ_1 的方程的右邊 $=3\cdot 1/\sqrt{3}\cdot 2+9-2\sqrt{3}=2\sqrt{3}+9-2\sqrt{3}=9=3\cdot 3=\ell_1$ 的方程的左邊。

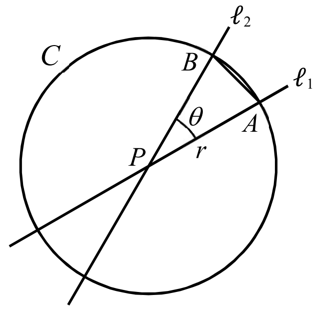
$\therefore$ P 的座標滿足 ℓ_1 的方程，換句話說 ℓ_1 通過 P。

此外，當 $x=2$ 及 $y=3$，

ℓ_2 的方程的右邊 $=\sqrt{3}\cdot 2-2\sqrt{3}+3=2\sqrt{3}-2\sqrt{3}+3=3=\ell_2$ 的方程的左邊。

$\therefore$ ℓ_2 也通過 P。

(c) $\because$ A、B 的 x 座標均大於 3，$\therefore$ A、B 均位於 C 的右半部份(如右圖所示)。



設 $\theta=\angle APB$。

由 ℓ_1 和 ℓ_2 的方程，得知 ℓ_1 和 ℓ_2 的傾斜角分別為 30° 和 60°。從而得知 $\theta=60°-30°=30°$。

$\therefore$ ΔPAB 的面積 $=\frac{1}{2}PA\cdot PB\sin\theta=\frac{1}{2}\cdot 5^2\cdot\sin 30°=\frac{25}{2}\cdot\frac{1}{2}=6.25$。

<u>Part I</u> Multiple choice questions. Choose the *best answer* for each question.

1. If $P=\{0, 1, 2\}$, $Q=\{0, 2, 3\}$, $R=\{1, 2, 3, 4\}$, then $P\cap(Q\cup R)=$

   A. $\{0, 1, 2\}$ B. $\{0, 1, 2, 3, 4\}$ C. $\{2\}$
   D. $\varnothing$ E. none of the above

2. Suppose the population of a city was 2,000,000 at the end of year 2017. If the population growth rate is 2% per year, at the end of year 2020 the population size will be approximately (select the closest answer):

   A. 2,100,000 B. 2,080,000 C. 2,140,000 D. 2,060,000 E. 2,120,000

3. $9x^2-3x-4y^2+2y=$

   A. $(3x-2y)(3x+2y+1)$ B. $(3x-2y)(3x+2y-1)$ C. $(3x+2y)(3x-2y+1)$
   D. $(2x+3y)(2x-3y-1)$ E. none of the above

4. If $\sqrt[n]{x}$ denotes the n^{th} root of x, which of the following is correct?

   A. $\sqrt[4]{17}<\sqrt{4.1}<\sqrt[3]{7.2}$ B. $\sqrt[3]{7.2}<\sqrt{4.1}<\sqrt[4]{17}$ C. $\sqrt[3]{7.2}<\sqrt[4]{17}<\sqrt{4.1}$
   D. $\sqrt[4]{17}<\sqrt[3]{7.2}<\sqrt{4.1}$ E. none of the above

5. If $|-3x-5|\le 4$, then

   A. $-\frac{1}{3}\le x\le 3$ B. $\frac{1}{3}\le x\le 3$ C. $-3\le x\le -\frac{1}{3}$
   D. $-3\le x\le \frac{1}{3}$ E. $x\le -3$ or $x\ge -\frac{1}{3}$

6. If $x^5=y^{13}$, which of the following is/are true?

   I. $\log_x y=\frac{13}{5}$ II. $\log_y x=\frac{13}{5}$ III. $\log_x y=\frac{5}{13}$ IV. $\log_y x=\frac{5}{13}$

   A. I only B. II only C. IV only
   D. II and III only E. I and IV only

7. If $\sqrt{x+7}-\sqrt{x}=2$, then $x=$

   A. $\frac{9}{16}$ B. $\frac{3}{16}$ C. $\frac{9}{4}$ D. $\frac{4}{9}$ E. $\frac{3}{4}$

8. Three distinct numbers are selected from the six numbers 0, 1, 3, 5, 6, 7 to form a 3-digit even number. How many ways are there? [Note: the number 013 is only 2-digit, not 3-digit.]

   A. 26 B. 36 C. 42 D. 45 E. none of the above

9. The constant term in the expansion of $\left(x^2+\frac{1}{x}\right)^{12}$ is

A. 495 B. 220 C. 66 D. 792 E. none of the above

10. Suppose $(a, 0)$ and $(0, -b)$ are the endpoints of a diameter of a circle. Which of the following points lie on the circle?

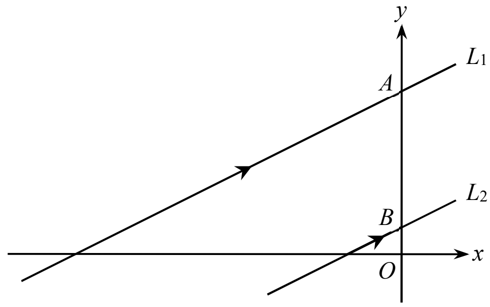
I. $(0, 0)$ II. (a, b) III. $(a, -b)$

A. I only B. III only C. I and II only
D. II and III only E. I and III only

11. If the graph of the parabola $y=(x+1)^2$ is shifted to the right by 4 units, and then shifted downward by 5 units, what is the equation of the resulting parabola?

A. $y=(x-3)^2-5$ B. $y=(x+5)^2-5$ C. $y=(x-5)^2+3$
D. $y=(x+3)^2-5$ E. $y=(x+4)^2-5$

12. In the right figure, two parallel lines L_1: $x-4y+12=0$ and L_2: $2x-my+4=0$ intersect the y-axis at points A and B respectively. Find the length of line segment AB.

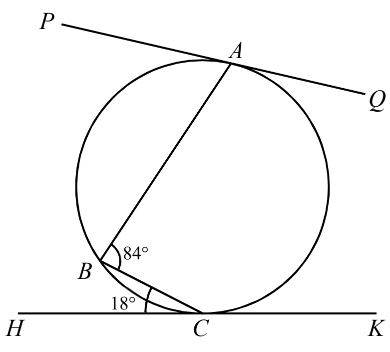


A. 4 B. 7/2
C. 5/2 D. 1
E. none of the above

13. In the arithmetic sequence $\{a_n\}_{n\geq 1}$, $a_2=22$ and $a_5=13$. How many positive terms does this sequence have?

A. 6 B. 7 C. 8 D. 9 E. 10

14. In the right figure, PQ and HK touch the circle at A and C respectively. Given that $\angle ABC=84°$ and $\angle BCH=18°$, find $\angle PAB$.



A. 45° B. 72°
C. 78° D. 85°
E. none of the above

15. If $\sin\theta-\cos\theta=-\frac{1}{5}$, and $\pi<\theta<2\pi$, what is the value of $\cos 2\theta$?

A. $\frac{7}{25}$ B. $-\frac{7}{25}$ C. $\pm\frac{7}{25}$ D. $-\frac{12}{25}$ E. $\frac{12}{25}$

<u>Part II</u> Problem-solving questions.

1. Mr. Tam found some computer disks of two brands in his drawer, in which 40% are of brand A. Among the disks, all brand A disks are usable, but only 40% of brand B disks are usable.
   (a) Find the probability that a randomly picked disk is usable. (4 marks)
   (b) Suppose there are totally 100 disks. If 2 disks are randomly picked, what is the probability that both of them are usable? (Express your answer as a fraction in lowest terms.) (4 marks)

2. A train from City A to City B was originally travelling at a speed of 40 km per hour. After two hours the train broke down, and it stopped for 45 minutes to repair its engine. And then it finished the remaining journey at a speed of 50 km per hour.
   (a) Let the distance between A and B be x km. In the above scenario, find an expression in terms of x, which gives the total time (in hours) required for the train to travel from A to B. (5 marks)
   (b) If the train was able to arrive on time, how far is it from A to B? (3 marks)

3. Given that the sequence $\{a_n\}_{n\geq 1}$ satisfies $a_{n+1} = a_n + \dfrac{2n+1}{n^2(n+1)^2}$ and $a_1 = 0$.
   (a) Find the values of a_2, a_3 and a_4. (3 marks)
   (b) Guess a formula for the general term of $\{a_n\}_{n\geq 1}$, and prove it by mathematical induction. (5 marks)

4. Given that one of the roots of the equation $a^{2x-4} - 5a^{x-2} + 6 = 0$ is 3, where a is a constant.
   (a) Find the value of a. (4 marks)
   (b) Find all the other root(s) of the equation. (4 marks)

5. Let $x^2 + y^2 - 4x - 6y - 12 = 0$, $3y = (3\tan 30°)x + 9 - 2\sqrt{3}$ and $y = (\tan 60°)x - (2\sqrt{3} - 3)$ be respectively the equations of a circle C, and two lines ℓ_1 and ℓ_2. Suppose A and B are respectively the intersecting points of ℓ_1 and ℓ_2 with C, where the x-coordinates of A and B are greater than 3.
   (a) Find the radius r and the coordinates of the center P of C. (2 marks)
   (b) Show that ℓ_1 and ℓ_2 pass through P. (3 marks)
   (c) Find the area of ΔPAB. (3 marks)

Suggested Answer

<u>Part I</u> Multiple choice questions.

Question Number	Best Answer
1	A
2	E
3	B
4	B
5	C
6	D
7	A
8	B
9	A
10	E
11	A
12	C
13	D
14	C
15	B

(Answers for Part II start from next page)

<u>Part II</u> Problem-solving questions.

1. (a) Probability of picking 1 usable disk $=40\%+(1-40\%)\times 40\%=0.64$.

   (b) Probability of picking 2 usable disks $=\dfrac{64}{100}\times\dfrac{63}{99}=\dfrac{112}{275}$.

2. (a) The train had travelled $40\times 2=80$ km in the first two hours. It had travelled $x-80$ km after stopping for 45 minutes, and arrived at B $\dfrac{x-80}{50}$ hours later. The whole journey took

$$2+\frac{45}{60}+\frac{x-80}{50}=\frac{23}{20}+\frac{x}{50}\text{ hours.}$$

   (b) The train was initially travelling at a speed of 40 km/hour. If no delay occurs, the train will arrive at B on time after $\dfrac{x}{40}$ hours. Therefore, we have

$$\frac{23}{20}+\frac{x}{50}=\frac{x}{40}\quad\Rightarrow x=230.$$

   That is, City A is 230 km from City B.

3. (a) $a_2=0+\dfrac{2\cdot 1+1}{1^2\cdot 2^2}=\dfrac{3}{4}$, $a_3=\dfrac{3}{4}+\dfrac{2\cdot 2+1}{2^2\cdot 3^2}=\dfrac{8}{9}$, and $a_4=\dfrac{8}{9}+\dfrac{2\cdot 3+1}{3^2\cdot 4^2}=\dfrac{15}{16}$.

   (b) From (a), we guess that $a_n=\dfrac{n^2-1}{n^2}$.

   Let $S(n)$ denote the statement "$a_n=\dfrac{n^2-1}{n^2}$".

   According to the given condition that $a_1=0$, $S(1)$ is true.

   Assume $S(k)$ is true for some positive integer k, that is, for this k, $a_k=\dfrac{k^2-1}{k^2}$.

   It follows from this assumption and the other given condition that

$$a_{k+1}=\frac{k^2-1}{k^2}+\frac{2k+1}{k^2(k+1)^2}=\frac{(k^2-1)(k+1)^2+2k+1}{k^2(k+1)^2}=\frac{k^2+2k}{(k+1)^2}=\frac{(k+1)^2-1}{(k+1)^2}.$$

   In other words, $S(k+1)$ is also true.
   By the Principle of Mathematical Induction, $S(n)$ is true for all positive integers n.

4. (a) Since $x=3$ is a root of the equation, we get $a^2-5a+6=0$, i.e. $(a-2)(a-3)=0$, and so $a=2$ or $a=3$.

   (b) When $a=2$, the original equation reduces to $2^{2(x-2)}-5\cdot 2^{x-2}+6=(2^{x-2}-2)(2^{x-2}-3)=0$. Then we have $2^{x-2}=2$ or $2^{x-2}=3$. That is, $x=3$ or $x=\log_2 3+2$.

   When $a=3$, the original equation reduces to $3^{2(x-2)}-5\cdot 3^{x-2}+6=(3^{x-2}-2)(3^{x-2}-3)=0$. Then we have $3^{x-2}=2$ or $3^{x-2}=3$. That is, $x=3$ or $x=\log_3 2+2$.
   so the remaining root is $\log_2 3+2$ and $\log_3 2+2$ 。

5. (a) By rewriting the equation of C as $(x-2)^2+(y-3)^2=5^2$, we obtain $P=(2, 3)$, $r=5$.

(b) It is known that $\tan 30° = 1/\sqrt{3}$ and $\tan 60° = \sqrt{3}$. When $x=2$ and $y=3$,

$$\text{RHS of } \ell_1 = 3\cdot 1/\sqrt{3}\cdot 2+9-2\sqrt{3} = 2\sqrt{3}+9-2\sqrt{3} = 9 = 3\cdot 3 = \text{LHS of } \ell_1.$$

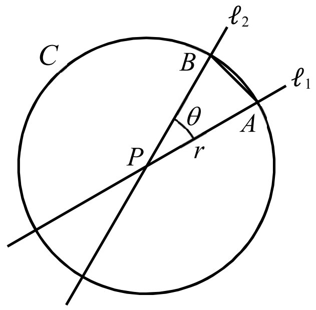
$\therefore$ the coordinates of P satisfy the equation of ℓ_1, in other words, ℓ_1 passes through P.

Furthermore, when $x=2$ and $y=3$,

$$\text{RHS of } \ell_2 = \sqrt{3}\cdot 2-2\sqrt{3}+3 = 2\sqrt{3}-2\sqrt{3}+3 = 3 = \text{LHS of } \ell_2.$$

$\therefore$ ℓ_2 also passes through P.

(c) $\because$ the x-coordinates of A and B are greater than 3, $\therefore$ both A and B lie on the right half of C (as shown in the right figure).



Let $\theta = \angle APB$.

From the equations of ℓ_1 and ℓ_2, we see that the angles of inclination of ℓ_1 and ℓ_2 are 30° and 60° respectively. Hence $\theta = 60° - 30° = 30°$.

$\therefore$ area of $\Delta PAB = \frac{1}{2} PA\cdot PB \sin\theta = \frac{1}{2}\cdot 5^2\cdot \sin 30° = \frac{25}{2}\cdot\frac{1}{2} = 6.25$.