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UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2025 年試題及參考答案
2025 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項:

1. 考生獲發文件如下:

   1.1 本考卷包括封面共22版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page −22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

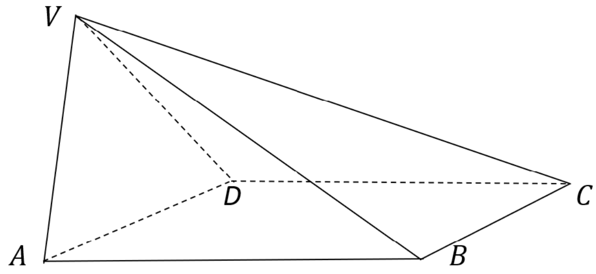
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1. 在如圖四棱錐 V-$ABCD$ 中，底面 $ABCD$ 是邊長為 a 的正方形。$\triangle VAD$ 是正三角形，且垂直於底面 $ABCD$。



(a) 設 M 為 AD 的中點，求線段 VM（用 a 表示）。（2 分）

(b) 求四棱錐 V-$ABCD$ 的體積。（4 分）

(c) 設 x 為 $\triangle VAD$ 與 $\triangle VDB$ 的二面角，求 $\tan x$。（7 分）

(d) 求四棱錐 V-$ABCD$ 的表面積。（7 分）

In the pyramid V-$ABCD$ shown in the figure, the base $ABCD$ is a square with a side length of a, $\triangle VAD$ is an equilateral triangle and is perpendicular to the base $ABCD$.

(a) Let M be the midpoint of AD. Find line segment VM (in terms of a). (2 marks)

(b) Find the volume of the pyramid V-$ABCD$. (4 marks)

(c) Let x be the dihedral angle between plane VAD and plane VDB. Find $\tan x$. (7 marks)

(d) Find the surface area of the pyramid V-$ABCD$. (7 marks)

2.

(a) 設 $f(x) = x^3 + 3x^2 - 4$。

(i) 求 $f(x) = 0$ 的根。 (2 分)

(ii) 求 $f'(x)$ 和 $f''(x)$。 (2 分)

(iii) 求 $f(x)$ 的局部極大值和局部極小值。 (4 分)

(iv) 求曲線 $y = f(x)$ 的拐點。 (1 分)

(v) 繪出曲線 $y = f(x)$ 的圖像，其中 $-3 \leq x \leq 1.2$。 (3 分)

(b) 求由曲線 $y = x^3 + 3x^2 - 4$ 及曲線 $y = x^3 - 3x + 2$ 所包圍的區域的面積。 (8 分)

(a) Let $f(x) = x^3 + 3x^2 - 4$.

(i) Find the roots of $f(x) = 0$. (2 marks)

(ii) Find $f'(x)$ and $f''(x)$ (2 marks)

(iii) Find the local maximum and local minimum values of $f(x)$. (4 marks)

(iv) Find the inflection point of the curve $y = f(x)$. (1 marks)

(v) Sketch the curve of $y = f(x)$, with $-3 \leq x \leq 1.2$. (3 marks)

(b) Find the area of the region bounded by the curves $y = x^3 + 3x^2 - 4$ and $y = x^3 - 3x + 2$. (8 marks)

3. 設橢圓 E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b>0)$ 的右焦點為 F_1，經過焦點 F_1 和點 $P(2,1)$ 的直線 F_1P 與橢圓 E 相交於 A 和 B 兩點，已知 $A(0,-1)$。

(a) 求直線 F_1P 的方程。 （4 分）

(b) 求橢圓 E 的兩個焦點 F_1 和 F_2 的坐標。 （4 分）

(c) 求橢圓 E 方程中 a,b 的值。 （6 分）

(d) 設直線 l 與直線 F_1P 平行，且與 y 軸交於點 $(0,m)$。求直線 l 與橢圓 E 相切時 m 的值。 （6 分）

Suppose the right focus of the ellipse E: $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ (a>b>0)$ is F_1. The line F_1P passing through the focus F_1 and the point $P(2,1)$ intersects the ellipse E at two points A and B. Given $A(0,-1)$.

(a) Find the equation of the line F_1P. (4 marks)

(b) Find the coordinates of the two foci F_1 and F_2 of the ellipse E. (4 marks)

(c) Find values of a and b in the equation of the ellipse E. (6 marks)

(d) Suppose that line l is parallel to line F_1P and intersects the y-axis at point $(0,m)$. Find the value of m when line l is tangent to the ellipse E. (6 marks)

4. 設 $i = \sqrt{-1}$。

(a) 設 $z = a + bi$，其中 a, b 為實數，且滿足 $b \neq 0$ 及 $z^3 = 8$，求 z。 (4 分)

(b) 對 (a) 中所得 z, 求 z^{2024}。 (4 分)

(c) 設 $w = \cos\theta + i\sin\theta$，其中 $0 < \theta < 2\pi$ 且 $\theta \neq \pi$。對任意正偶數 n，証明

$$1 + w + w^2 + w^3 + \cdots + w^n = w^{\frac{n}{2}} \frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

(4 分)

(d) 求以下方程在 $0 < \theta < 2\pi$ 內的所有實根

$$\sin\theta + \sin(2\theta) + \sin(3\theta) + \sin(4\theta) + \sin(5\theta) + \sin(6\theta) = 0.$$

(8 分)

Let $i = \sqrt{-1}$.

(a) Suppose $z = a + bi$, where a, b are real numbers satisfying $b \neq 0$ and $z^3 = 8$, find z. (4 marks)

(b) For z obtained from (a), find z^{2024}. (4 marks)

(c) Let $w = \cos\theta + i\sin\theta$, where $0 < \theta < 2\pi$ and $\theta \neq \pi$. For any even positive integer n, show that

$$1 + w + w^2 + w^3 + \cdots + w^n = w^{\frac{n}{2}} \frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

(4 marks)

(d) Find all real roots θ with $0 < \theta < 2\pi$, for the following equation,

$$\sin\theta + \sin(2\theta) + \sin(3\theta) + \sin(4\theta) + \sin(5\theta) + \sin(6\theta) = 0.$$

(8 marks)

5.

(a) 已知矩陣 $C = \begin{bmatrix} a+b & b+c & c+a \\ a-b & b-c & c-a \\ c & a & b \end{bmatrix}$。

(i) 求行列式 $|C|$。 (4 分)

(ii) 求方程 $|C| = 0$ 的所有實數解組 (a, b, c)。 (4 分)

(b) 設 k、p 和 q 為常數，及 (E) 是以 x、y 和 z 為未知量的方程組:

$$(E): \begin{cases} kx + 2y - z = p \\ ky + z = q \\ kx + 3y = 6 \end{cases}$$

(i) 求 k 的取值範圍，使得 (E) 有唯一解。 (4 分)

(ii) 當 (E) 有多於一個解時，求 p 和 q 的關係，並寫出方程 (E) 的解。 (8 分)

(a) Let the matrix $C = \begin{bmatrix} a+b & b+c & c+a \\ a-b & b-c & c-a \\ c & a & b \end{bmatrix}$.

(i) Calculate the determinant $|C|$。 (4 marks)

(ii) Find all real solutions (a, b, c) such that $|C| = 0$。 (4 marks)

(b) Let k, p and q be constants, and let (E) be a system of equations with unknowns x, y and z:

$$(E): \begin{cases} kx + 2y - z = p \\ ky + z = q \\ kx + 3y = 6 \end{cases}$$

(i) Find the range of k, such that (E) has a unique solution. (4 marks)

(ii) When (E) has more than one solution, find the relationship between p and q, and write down the general solution of (E). (8 marks)

参考答案

1. (a) $\because \triangle VAD$ 為正三角形，$AD=a$，且 M 為 AD 中點，

$\therefore VM \perp AD$。

$\therefore VM = \dfrac{\sqrt{3}}{2}a$。

(b) $\because$ 平面 $\triangle VAD$ 垂直於底面 $ABCD$，且 $VM \perp AD$，

$\therefore VM \perp ABCD$，則 VM 為四棱錐 $V-ABCD$ 的高，

$\therefore V\text{-}ABCD$ 的體積 $=\dfrac{1}{3}VM\cdot AB\cdot AD=\dfrac{\sqrt{3}}{6}a^3$。

(c) 設 P 為 VD 的中點，連接線段 AP，BP。

$\because AB\perp AD$，$AB\perp VM$，

$\therefore AB\perp \triangle VAD$，

$\therefore AB\perp AP$，$AB\perp AV$，

$\therefore \triangle APB$ 為直角三角形。

$\because \triangle AVB$ 中，$AV=AB$，

$\therefore VB=\sqrt{2}a$。

$\because$ 在正方形 $ABCD$ 中，對角線 $BD=\sqrt{2}a$，

$\therefore \triangle VBD$ 為等腰三角形。

$\because P$ 為 VD 的中點，

$\therefore BP\perp VD$，

$\therefore \angle APB=x$，$\tan x=\dfrac{AB}{AP}=\dfrac{2\sqrt{3}}{3}$。

(d) $\because$ 在 $\triangle VBC$ 中，$VB=VC=\sqrt{2}a$，$BC=a$，

$\therefore \triangle VBC$ 的面積為 $\dfrac{\sqrt{7}}{4}a^2$。

$\because AB=CD=VA=VD=a$，$VA\perp AB$，$VD\perp CD$，

$\therefore \triangle VAB$ 全等於 $\triangle VDC$，

$\therefore \triangle VAB$ 和 $\triangle VDC$ 的面積為 $\dfrac{1}{2}a^2$。

$\because \triangle VAD$ 的面積為 $\dfrac{\sqrt{3}}{4}a^2$，正方形 $ABCD$ 的面積為 a^2，

$\therefore$ 四棱錐 $V\text{-}ABCD$ 的表面積為正方形 $ABCD$ 和 $\triangle VAB$，$\triangle VAD$，$\triangle VDC$，$\triangle VBC$ 的面積之和 $=2+\dfrac{\sqrt{3}+\sqrt{7}}{4}a^2$。

2. (a) (i) 由 $f(x)=x^3-x^2+4x^2-4=(x-1)(x+2)^2=0$，可得 $x=1$ 或 $x=-2$。

(ii) $f'(x)=3x^2+6x$，$f''(x)=6x+6$。

(iii) 由 $f_\prime(x)=0$ 得 $x=0$ 或 $x=-2$。

當 $x<-2$，$f'(x)>0$，故 $f(x)$ 是遞增的。

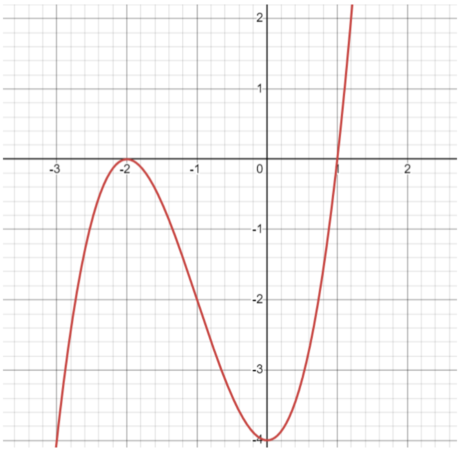
當 $-2<x<0$，$f'(x)<0$，故 $f(x)$ 是遞減的。

當 $x>0$，$f'(x)>0$，故 $f(x)$ 是遞增的。

因此，$f(-2)=0$ 是一局部極大值，$f(0)=-4$ 是一局部極小值。

(iv) 由 $f''(x)=0$ 得 $x=-1$。當 $x<-1$，$f''(x)<0$；當 $x>-1$，$f''(x)>0$。因此，$(-1,-2)$ 是曲線 $y=f(x)$ 的拐點。

(v)



(b) 解 $\begin{cases} y=x^3+3x^2-4 \\ y=x^3-3x+2 \end{cases}$，得 $x=-2$ 或 $x=1$。

當 $-2<x<1$，曲線 $y=x^3-3x+2$ 在曲線 $y=x^3+3x^2-4$ 之上。

因此，所求面積為

$$\begin{aligned}\int_{-2}^{1}(x^3-3x+2)-(x^3+3x^2-4)dx &= -3\int_{-2}^{1}x^2+x-2\,dx \\ &= -3\left[\frac{x^3}{3}+\frac{x^2}{2}-2x\right]_{-2}^{1}=\frac{27}{2}\text{。}\end{aligned}$$

3. (a) $\because F_1P$ 經過 $P(2,1)$ 和 $A(0,-1)$

$\therefore m=\frac{1-(-1)}{2-0}=1$

$\therefore b=-1$

$\therefore y=x-1$

(b) $\because F_1$ 為橢圓 E 的右焦點，且在直線 F_1P 上。

$\therefore$ 令 y=0

$\therefore$ x=1, 即 $c=1$

$\therefore F_1(1,0), F_2(-1,0)$.

(c) $\because |AF_1|+|AF_2|=2\sqrt{2}=2a$

$\therefore a=\sqrt{2}$,

$\because c=1$,

$\therefore b=1$.

可得 E 的方程为 $\frac{x^2}{2}+y^2=1$.

(d) 設直線 l 方程為 $y=x+m$。由直線 l 和橢圓 E 相切可得：

$$\begin{cases} y=x+m \\ \dfrac{x^2}{2}+y^2=1 \end{cases}。$$

由方程組得 $3x^2+4mx+2m^2-2=0$。令 $\triangle=16m^2-24(m^2-1)=0$，可得 $m^2=3$，即 $m=\pm\sqrt{3}$。

4. (a) 由題可知，$z^3-8=0$，即 $(z-2)(z^2+2z+4)=0$。由於 $z=a+bi$，$b\neq 0$，可得 $z=-1+\sqrt{3}i$ 或 $z=-1-\sqrt{3}i$。

(b) 當 $z=-1+\sqrt{3}i, z=-2(\cos(-\frac{\pi}{3}), i\sin(-\frac{\pi}{3}))$，故

$$\begin{aligned} z^{2024} &= (-2)^{2024}(\cos(-\frac{\pi}{3}), i\sin(-\frac{\pi}{3}))^{2024} \\ &= 2^{2024}(\cos(-\frac{2\pi}{3}), i\sin(-\frac{2\pi}{3})) \\ &= -2^{2024}(\frac{1}{2}+\frac{\sqrt{3}}{2}i) \\ &= -2^{2023}(1+\sqrt{3}i). \end{aligned}$$

當 $z=-1-\sqrt{3}i, z=-2(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3})$，故

$$\begin{aligned} z^{2024} &= 2^{2024}(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}) \\ &= 2^{2023}(-1+\sqrt{3}i). \end{aligned}$$

(c) 對任意正偶數 n，

$$1+w+w^2+w^3+\cdots+w^n=\frac{w^{n+1}-1}{w-1}=\frac{w^{\frac{n+1}{2}}}{w^{\frac{1}{2}}}\frac{w^{\frac{n+1}{2}}-w^{-\frac{n+1}{2}}}{w^{\frac{1}{2}}-w^{-\frac{1}{2}}}=w^{\frac{n}{2}}\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

(d) 由於 $0<\theta<2\pi$ 及 $\theta\neq\pi$，$\sin\theta/2\neq 0$，則方程

$$\sin\theta+\sin(2\theta)+\sin(3\theta)+\sin\ (4\theta)+\sin\ (5\theta)+\sin\ (6\theta)=0$$

等價於方程

$$\sin\frac{6\theta}{2}\sin\frac{7\theta}{2}=0,$$

其解為: $\theta=\frac{k\pi}{3}$ $(k=1,2,3,4,5)$，或 $\frac{m\pi}{7}$ $(m=2,4,6,8,10,12)$。

5.

(a)(i) $C=\begin{bmatrix} a+b & b+c & c+a \\ a-b & b-c & c-a \\ c & a & b \end{bmatrix}$

$$\begin{aligned}
|C| &= \begin{vmatrix} a+b & b+c & a+c \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} \\
&= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2b-a-c & c-2a+b \\ 0 & a-c & b-c \end{vmatrix} \\
&= (a+b+c)(2a^2+2b^2+2c^2-2ab-2bc-2ac)
\end{aligned}$$

(ii) 上式 =

$(a+b+c)[(a-b)^2+(b-c)^2+(a-c)^2]=0,$

因此 $(a,b,c)=(r,s,-r-s),$

或 $(a,b,c)=(r,r,r),\quad \forall r,s\in\mathbb{R}.$

(b)(i) 方程 (E) 有唯一解當且僅當

$$|C| = \begin{vmatrix} k & 2 & -1 \\ 0 & k & 1 \\ k & 3 & 0 \end{vmatrix} = \begin{vmatrix} k & 2 & -1 \\ 0 & k & 1 \\ 0 & 1 & 1 \end{vmatrix} = k \begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} = k(k-1) \neq 0,$$

即 $k \neq 0$ 和$k \neq 1$.

(ii) 當 $k = 0, (E)$ 為 $\begin{cases} 2y - z = p \\ z = q \\ 3y = 6 \end{cases}$,

p與q需同時滿足 $p + q = 4$.

此時，通解為 $x = t, y = 2, z = q$.

當 $k = 1, (E)$為 $\begin{cases} x + 2y - z = p \\ y + z = q \\ x + 3y = 6 \end{cases}$,

p與q需同時滿足 $p + q = 6$.

此時，通解為 $x = 6 - 3t, y = t, z = q - t, \quad \forall t \in \mathbb{R}$.

Suggested Answers:

1. (a) $\because \triangle VAD$ is an equilateral triangle, $AD = a$, and M is the midpoint of AD,

$\therefore VM \perp AD$.

$\therefore VM = \dfrac{\sqrt{3}}{2}a$.

(b) $\because$ plane $\triangle VAD$ is perpendicular to the base $ABCD$, and $VM \perp AD$,

$\therefore VM \perp ABCD$, then VM is the height of the tetrahedron $V - ABCD$,

$\therefore$ The volume of $V - ABCD = \dfrac{1}{3}VM \cdot AB \cdot AD = \dfrac{\sqrt{3}}{6}a^3$.

(c) Let P be the midpoint of VD and connect line segments AP and BP.

$\because AB \perp AD$, $AB \perp VM$,

$\therefore AB \perp \triangle VAD$,

$\therefore AB \perp AP$, $AB \perp AV$,

$\therefore \triangle APB$ is a right triangle.

$\because \triangle AVB$, $AV = AB$,

$\therefore VB = \sqrt{2}a$.

$\because$ in the square $ABCD$, the diagonal $BD = \sqrt{2}a$,

$\therefore \triangle VBD$ is an isosceles triangle.

$\because P$ is the midpoint of VD,

$\therefore BP \perp VD$,

$\therefore \angle APB = x$, tan $x = \dfrac{AB}{AP} = \dfrac{2\sqrt{3}}{3}$.

(d) $\because$ In $\triangle VBC$, $VB = VC = \sqrt{2}a$, $BC = a$,

$\therefore$ The area of $\triangle VBC$ is $\dfrac{\sqrt{7}}{4}a^2$.

$\because AB = CD = VA = VD = a$, $VA \perp AB$, $VD \perp CD$,

$\therefore \triangle VAB$ is congruent to $\triangle VDC$,

$\therefore$ The area of both $\triangle VAB$ and $\triangle VDC$ is $\dfrac{1}{2}a^2$.

$\because \triangle VAD$ has an area of $\dfrac{\sqrt{3}}{4}a^2$, and the area of the square $ABCD$ is a^2,

$\therefore$ The surface area of the tetrahedron V-$ABCD$ is the sum of the areas of the square $ABCD$ and $\triangle VAB$, $\triangle VAD$, $\triangle VDC$, $\triangle VBC = 2 + \dfrac{\sqrt{3}+\sqrt{7}}{4}a^2$.

2. (a) (i) From $f(x) = x^3 - x^2 + 4x^2 - 4 = (x-1)(x+2)^2 = 0$, we have $x = 1$ or $x = -2$.

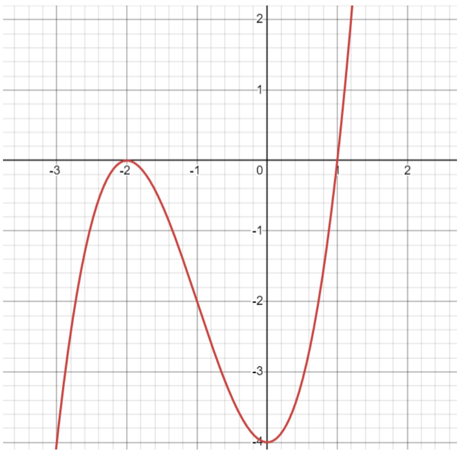
(ii) $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$.

(iii) From $f'(x) = 0$, we have $x = 0$ or $x = -2$.

When $x < -2$, $f'(x) > 0$, so $f(x)$ is increasing.
When $-2 < x < 0$, $f'(x) < 0$, so $f(x)$ is decreasing.
When $x > 0$, $f'(x) > 0$, so $f(x)$ is increasing.
Hence, $f(-2) = 0$ is a local maximum value, $f(0) = -4$ is a local minimum value.

(iv) From $f''(x) = 0$, we have $x = -1$. When $x < -1$, $f''(x) < 0$; When $x > -1$, $f''(x) > 0$.
Hence, $(-1, -2)$ is the inflection point of the curve $y = f(x)$.

(v)



(b) Solving $\begin{cases} y = x^3 + 3x^2 - 4 \\ y = x^3 - 3x + 2 \end{cases}$, we obtain $x = -2$ or $x = 1$.
When $-2 < x < 1$, the curve $y = x^3 - 3x + 2$ is above the curve $y = x^3 + 3x^2 - 4$.
Hence, the required area is

$$\begin{aligned} \int_{-2}^{1} (x^3 - 3x + 2) - (x^3 + 3x^2 - 4)dx &= -3\int_{-2}^{1} x^2 + x - 2\,dx \\ &= -3\left[\frac{x^3}{3} + \frac{x^2}{2} - 2x\right]_{-2}^{1} = \frac{27}{2}. \end{aligned}$$

3. (a) $\because F_1P$ passes through $P(2, 1)$ and $A(0, -1)$
$\therefore m = \frac{1-(-1)}{2-0} = 1$
$\therefore b = -1$
$\therefore y = x - 1$

(b) $\because F_1$is the right focus of the ellipse of E and is on the line F_1P.
$\therefore$ Let y=0
$\therefore$ x=1, that is $c = 1$

$\therefore F_1(1,0), F_2(-1,0)$.

(c) $\because |AF_1| + |AF_2| = 2\sqrt{2} = 2a$

$\therefore a = \sqrt{2}$,

$\because c = 1$,

$\therefore b = 1$,

and the equation of the ellipse E is $\frac{x^2}{2} + y^2 = 1$.

(d) Suppose the equation of the line l is $y = x + m$。From the tangency of the line l and the ellipse E, we have:

$$\begin{cases} y = x + m \\ \dfrac{x^2}{2} + y^2 = 1 \end{cases}.$$

From the equation system, we get $3x^2+4mx+2m^2-2 = 0$. Let $\triangle = 16m^2-24(m^2-1) = 0$. We get $m^2 = 3$, that is, $m = \pm\sqrt{3}$.

4. (a) From the question, it is known that $z^3 - 8 = 0$, that is, $(z-2)(z^2+2z+4) = 0$. Since $z = a + bi, b \neq 0$, we obtain

$$z = -1 + \sqrt{3}i, \text{ or } z = -1 - \sqrt{3}i.$$

(b) While $z = -1 + \sqrt{3}i$, $z = -2(\cos(-\frac{\pi}{3}), i\sin(-\frac{\pi}{3}))$, hence

$$\begin{aligned} z^{2024} &= (-2)^{2024}(\cos(-\frac{\pi}{3}), i\sin(-\frac{\pi}{3}))^{2024} \\ &= 2^{2024}(\cos(-\frac{2\pi}{3}), i\sin(-\frac{2\pi}{3})) \\ &= -2^{2024}(\frac{1}{2} + \frac{\sqrt{3}}{2}i) \\ &= -2^{2023}(1+\sqrt{3}i). \end{aligned}$$

While $z = -1 - \sqrt{3}i$, $z = -2(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$, hence

$$\begin{aligned} z^{2024} &= 2^{2024}(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}) \\ &= 2^{2023}(-1+\sqrt{3}i). \end{aligned}$$

(c) For any positive integer n,

$$1 + w + w^2 + w^3 + \cdots + w^n = \frac{w^{n+1}-1}{w-1} = \frac{w^{\frac{n+1}{2}}}{w^{\frac{1}{2}}}\frac{w^{\frac{n+1}{2}} - w^{-\frac{n+1}{2}}}{w^{\frac{1}{2}} - w^{-\frac{1}{2}}} = w^{\frac{n}{2}}\frac{\sin\frac{(n+1)\theta}{2}}{\sin\frac{\theta}{2}}.$$

(d) Since $0 < \theta < 2\pi$ and $\theta \neq \pi$, so $\sin\theta/2 \neq 0$. Then the equation

$$\sin\theta + \sin(2\theta) + \sin(3\theta) + \sin\ (4\theta) + \sin\ (5\theta) + \sin\ (6\theta) = 0$$

is equivalent to the equation

$$\sin\frac{6\theta}{2}\sin\frac{7\theta}{2} = 0.$$

Thus the solutions are $\theta = \frac{k\pi}{3}$ $(k = 1, 2, 3, 4, 5)$, or $\frac{m\pi}{7}(m = 2, 4, 6, 8, 10, 12)$.

5.

(a)(i) $C = \begin{bmatrix} a+b & b+c & c+a \\ a-b & b-c & c-a \\ c & a & b \end{bmatrix}$

$$\begin{aligned}
|C| &= \begin{vmatrix} a+b & b+c & a+c \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} = \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} \\
&= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ a-b & b-c & c-a \\ c & a & b \end{vmatrix} = (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 0 & 2b-a-c & c-2a+b \\ 0 & a-c & b-c \end{vmatrix} \\
&= (a+b+c)(2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac)
\end{aligned}$$

(ii)The above equation is equal to

$$(a+b+c)[(a-b)^2+(b-c)^2+(a-c)^2]=0,$$

Hence $(a,b,c)=(r,s,-r-s)$,

or $(a,b,c)=(r,r,r), \quad \forall r,s\in\mathbb{R}$.

(b)(i) Equation (E) has a unique solution if and only if

$$|C| = \begin{vmatrix} k & 2 & -1 \\ 0 & k & 1 \\ k & 3 & 0 \end{vmatrix} = \begin{vmatrix} k & 2 & -1 \\ 0 & k & 1 \\ 0 & 1 & 1 \end{vmatrix} = k\begin{vmatrix} k & 1 \\ 1 & 1 \end{vmatrix} = k(k-1)\neq 0,$$

i.e., $k\neq 0$ and $k\neq 1$.

(ii) When $k=0$, (E) becomes $\begin{cases} 2y-z=p \\ z=q \\ 3y=6 \end{cases}$, the relationship between p and q is $p+q=4$, and the general solution of (E) is $x=t, y=2, z=q$.

When $k=1$, (E) becomes $\begin{cases} x+2y-z=p \\ y+z=q \\ x+3y=6 \end{cases}$, the relationship between p and q is $p+q=6$, and the general solution of (E) is $x=6-3t, y=t, z=q-t, \forall t\in\mathbb{R}$..