澳門大學
UNIVERSIDADE DE MACAU
UNIVERSITY OF MACAU

澳門理工大學
Universidade Politécnica de Macau
Macao Polytechnic University

澳門旅遊學院
INSTITUTO DE FORMAÇÃO TURÍSTICA DE MACAU
Macao Institute for Tourism Studies

澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2022 年試題及參考答案
2022 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共22版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

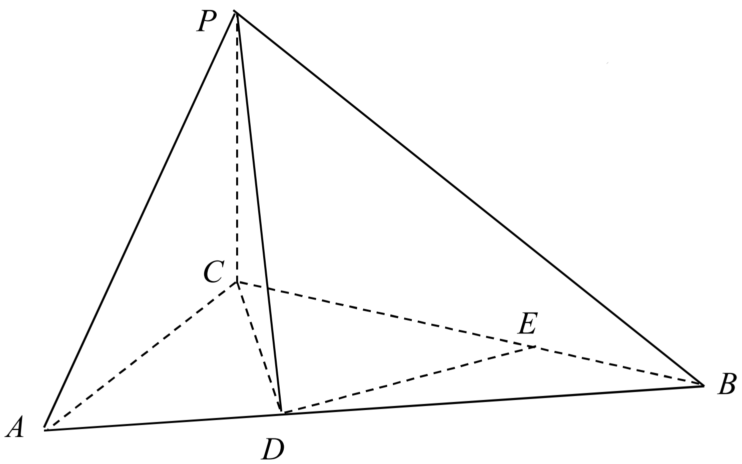
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖所示，在三棱錐 P-ABC 中，PC 垂直 ABC，$|PC| = 3$，$\angle BCA = \frac{\pi}{2}$，

D 和 E 分別是 AB 和 BC 上的點，且 $|CD| = |DE| = \sqrt{2}$，$|CE| = 2$，$|EB| = 1$。

(a) 求 $|AC|$。[提示: 設 M 為 CE 的中點。] (6 分)

(b) (i) 證明 $\angle CDE = \frac{\pi}{2}$。 (2 分)

(ii) 求 $\angle CDB$，答案以 $\cos^{-1}$ 表示。 (5 分)

(c) 求二面角 P-AB-C，答案以 $\tan^{-1}$ 表示。 (7 分)

[提示: 設 X 為 AB 上一點，使得 CX 與 AB 垂直。證明 PX 與 AB 垂直。]

As shown in the above figure, P-ABC is a triangular pyramid, PC is perpendicular to ABC, $|PC| = 3$, $\angle BCA = \frac{\pi}{2}$. The points D and E are on AB and BC, respectively, and $|CD| = |DE| = \sqrt{2}$, $|CE| = 2, |EB| = 1$.

(a) Find $|AC|$. [Hint. Let M be the midpoint of CE.] (6 marks)

(b) (i) Show that $\angle CDE = \frac{\pi}{2}$. (2 marks)

(ii) Find $\angle CDB$. Express your answer in terms of $\cos^{-1}$. (5 marks)

(c) Find the dihedral angle P-AB-C. Express you answer in terms of $\tan^{-1}$.

[Hint: Let X be the point on AB such that CX and AB are perpendicular.

Show that PX and AB are perpendicular.] (7 marks)

2. (a) 已知函數 $f(x) = x^3 - 3x^2 + 2$，且 $f(1) = 0$。

(i) 求方程 $f(x) = 0$ 的所有解。 (3 分)

(ii) 求 $f'(x)$ 及 $f''(x)$。 (2 分)

(iii) 求 $f(x)$ 的局部極大值和局部極小值。 (3 分)

(iv) 求曲線 $y = f(x)$ 的拐點。 (2 分)

(v) 運用 (i) – (iv) 的結果，繪出曲線 $y = f(x)$。 (3 分)

(b) 求由曲線 $y = x^2 - 8x + 24$ 及 $y = 8x - x^2$ 所包圍的區域的面積。 (7 分)

(a) Given function $f(x) = x^3 - 3x^2 + 2$, and that $f(1) = 0$.

(i) Find all the solution(s) of the equation $f(x) = 0$. (3 marks)

(ii) Find $f'(x)$ and $f''(x)$. (2 marks)

(iii) Find the local maximum and local minimum values of $f(x)$. (3 marks)

(iv) Find the inflection point(s) of the curve $y = f(x)$. (2 marks)

(v) Using the results in (i) – (iv), sketch the curve $y = f(x)$. (3 marks)

(b) Find the area of the region bounded by the curves $y = x^2 - 8x + 24$ and $y = 8x - x^2$. (7 marks)

3. 已知定點 $A(-1,0)$ 和 $B(1,0)$。曲線 C 上任一點 $N(x,y)$ 都有 $\overrightarrow{AN}\cdot\overrightarrow{AB}=\left|\overrightarrow{BN}\right|\cdot\left|\overrightarrow{AB}\right|$。

(a) 證明 C 是拋物線 $y^2=4x$。 (4 分)

(b) 若直線 $y=ax+b$ 與 C 相切，證明 $ab=1$。 (4 分)

(c) 設 $m>0$。

(i) 除原點以外，求直線 $L_1: y=mx$ 與 C 的交點 P。答案以 m 表示。 (2 分)

(ii) 求曲線 C 在 P 的切線 L_2 的斜率。答案以 m 表示。 (4 分)

(iii) 求 m 的值使得 L_1 與 L_2 的夾角為 $\tan^{-1}\frac{1}{4}$。 (6 分)

Given fixed points $A(-1,0)$ and $B(1,0)$. Any point $N(x,y)$ on the curve C satisfies $\overrightarrow{AN}\cdot\overrightarrow{AB}=\left|\overrightarrow{BN}\right|\cdot\left|\overrightarrow{AB}\right|$.

(a) Show that C is the parabola $y^2=4x$. (4 marks)

(b) Suppose that the straight line $y=ax+b$ is tangent with C. Show that $ab=1$. (4 marks)

(c) Suppose $m>0$.

(i) Find, besides the origin, the intersection point P of the straight line $L_1: y=mx$ and C. Express your answer in terms of m. (2 marks)

(ii) Find the slope of the tangent line L_2 of C at P. Express your answer in terms of m. (4 marks)

(iii) Find the value(s) of m such that the angle between L_1 and L_2 is $\tan^{-1}\frac{1}{4}$. (6 marks)

4. 設 $i = \sqrt{-1}$。

(a) (i) 設 $z_1 = 3 + 5i$ 及 $z_2 = 5 + i$。若 $z = x + yi$ 滿足 $|z - z_1| = |z - z_2|$，

其中 x 和 y 為實數，求 x 和 y 的關係式。 (4 分)

(ii) 在阿根圖中繪出 z_1，z_2 和 z 的軌跡。 (2 分)

(iii) 求 $|z - z_1|$ 的最小值。 (2 分)

(b) 設 $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$。

(i) 證明 $\omega^7 = 1$，並推導出 $1 + \omega + \omega^2 + \cdots + \omega^6 = 0$。 (4 分)

(ii) 證明 $\omega^n + \omega^{-n} = 2\cos\frac{2n\pi}{7}$，$n = 1,2,3,....$ 。 (2 分)

(ii) 用 (i) 和 (ii) 的結果，求 $\left(\cos\frac{2\pi}{7}\right)^2 + \left(\cos\frac{4\pi}{7}\right)^2 + \left(\cos\frac{6\pi}{7}\right)^2$ 的值。 (6 分)

Let $i = \sqrt{-1}$.

(a) (i) Let $z_1 = 3 + 5i$ and $z_2 = 5 + i$. Suppose $z = x + yi$ satisfies $|z - z_1| = |z - z_2|$,

where *x* and *y* are real. Find a relation between *x* and *y*. (4 marks)

(ii) In the Argand diagram, plot the points z_1, z_2 and the locus of *z*. (2 marks)

(iii) Find the minimum value of $|z - z_1|$. (2 marks)

(b) Let $\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$.

(i) Show that $\omega^7 = 1$. Deduce that $1 + \omega + \omega^2 + \cdots + \omega^6 = 0$. (4 marks)

(ii) Show that $\omega^n + \omega^{-n} = 2\cos\frac{2n\pi}{7}$, $n = 1,2,3,.....$ (2 marks)

(ii) Using the results in (i) and (ii), find the value of

$\left(\cos\frac{2\pi}{7}\right)^2 + \left(\cos\frac{4\pi}{7}\right)^2 + \left(\cos\frac{6\pi}{7}\right)^2$. (6 marks)

5. (a) 因式分解行列式 $\begin{vmatrix} a & b+c & b^2+c^2 \\ b & a+c & a^2+c^2 \\ c & a+b & a^2+b^2 \end{vmatrix}$。 (8 分)

(b) 已知以 x、y 和 z 為未知量的方程組:

$$(E): \begin{cases} x + \ y \ + pz = 1 \\ px + \ y + \ z = p \\ x + py + \ z = q \end{cases},$$

其中 p 和 q 為常數。

(i) 求 p 的取值範圍，使得 (E) 有唯一解。 (4 分)

(ii) 對使得 (E) 有多於一個解的 p 及 q 的值，求 (E) 的通解。 (8 分)

(a) Factorize the determinant $\begin{vmatrix} a & b+c & b^2+c^2 \\ b & a+c & a^2+c^2 \\ c & a+b & a^2+b^2 \end{vmatrix}$. (8 marks)

(b) Given the system of equations with unknowns x, y and z:

$$(E): \begin{cases} x + \ y \ + pz = 1 \\ px + \ y + \ z = p \\ x + py + \ z = q \end{cases},$$

where p and q are constants.

(i) Find the range of p such that (E) has a unique solution. (4 marks)

(ii) Find the general solution of (E) for those values of p and q such that (E) has more than one solution. (8 marks)

參考答案：

1. (a) 設 M 為 CE 的中點。因 $|CD| = |DE|$，ΔCDE 是一等腰三角形，故 $DM \perp CE$。

已知 $\angle BCA = \frac{\pi}{2}$，故 ΔABC 和 ΔDBM 相似，從而得 $\frac{|AC|}{|CB|} = \frac{|DM|}{|MB|}$。

因 $|DM| = \sqrt{|DE|^2 - |ME|^2} = 1$，故 $|AC| = \frac{|DM|}{|MB|}|CB| = \frac{3}{2}$。

(b) (i) 因 $|CE|^2 = 4 = |CD|^2 + |DE|^2$，故 $\angle CDE = \frac{\pi}{2}$。

(ii) 由 (a) 的解中，有 $|DM| = 1$。故 $|DB| = \sqrt{|DM|^2 + |MB|^2} = \sqrt{5}$。因此，

$$\angle CDB = \cos^{-1}\left(\frac{|CD|^2+|BD|^2-|CB|^2}{2|CD||BD|}\right) = \cos^{-1}\left(\frac{2+5-9}{2\sqrt{2}\sqrt{5}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right)\text{。}$$

(c) 設 X 為 AB 上一點，使得 $CX \perp AB$。連同 $PC\perp AB$ (因 $PC \perp ABC$)，得

$PX \perp AB$。因而得知二面角 P-AB-C 與 $\angle PXC$ 相等及 $\angle PXC = \tan^{-1}\frac{|PC|}{|CX|}$。

現考慮 ΔABC 的面積以求$|CX|$。從 $\frac{1}{2}|AC||BC| = \frac{1}{2}|AB||CX|$ ，得

$|CX| = \frac{|AC||BC|}{|AB|} = \frac{(\frac{3}{2})3}{\sqrt{\left(\frac{3}{2}\right)^2+3^2}} = \frac{3}{\sqrt{5}}$。因此， 二面角 P-AB-C 是 $\tan^{-1}\frac{3}{\frac{3}{\sqrt{5}}} = \tan^{-1}\sqrt{5}$。

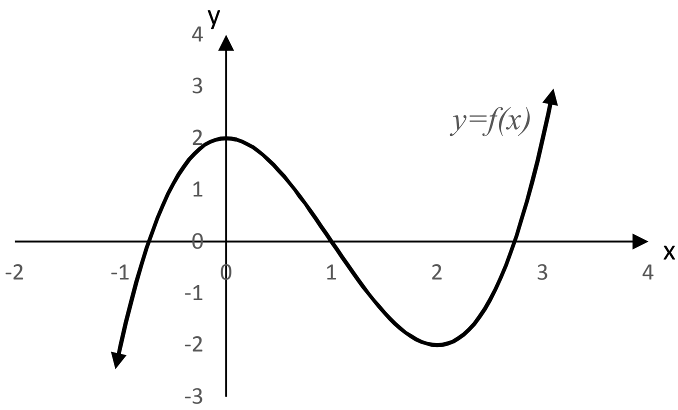
2. (a) (i) 因 $f(1)=0$，故 $x-1$是 $f(x)$ 的因式。計算得 $f(x)=(x-1)(x^2-2x-2)$。
故 $f(x)=0 \Leftrightarrow x=1$ 或 $x^2-2x-2=0 \Leftrightarrow x=1$ 或 $x=1\pm\sqrt{3}$。

(ii) $f'(x)=3x^2-6x$ ，$f''(x)=6x-6$。

(iii) $f'(x)=0 \Leftrightarrow x=0$ 或 $x=2$。
當 $x<0$，$f'(x)>0$ ，故 $f(x)$ 是遞增的。
當 $0<x<2$，$f'(x)<0$ ，故 $f(x)$ 是遞減的。
當 $2<x$，$f'(x)>0$ ，故 $f(x)$ 是遞增的。
因此，$f(0)=2$ 是一局部極大值，$f(2)=-2$ 是一局部極小值。

(iv) $f''(x)=0 \Leftrightarrow x=1$。當 $x<1$，$f''(x)<0$; 當 $x>1$，$f''(x)>0$。
因此，(1,0)是曲線 $y=f(x)$ 的拐點。

(v)



(b) 解 $\begin{cases} y=x^2-8x+24 \\ y=8x-x^2 \end{cases}$ ，得 $x=2$ 或 $x=6$。

當 $2<x<6$ ，曲線 $y=8x-x^2$ 在曲線 $y=x^2-8x+24$ 之上。

因此，所求面積為

$$\int_2^6 (8x-x^2)-(x^2-8x+24)\,dx = \int_2^6 -2x^2+16x-24\,dx$$

$$= \left[-\frac{2}{3}x^3+8x^2-24x\right]_2^6$$

$$= \frac{64}{3}。$$

3. (a) 因 $\overrightarrow{AN}=(x+1,y)$ ，$\overrightarrow{AB}=(2,0)$ 及 $\overrightarrow{BN}=(x-1,y)$ ，故

$$\overrightarrow{AN}\cdot\overrightarrow{AB}=\left|\overrightarrow{AB}\right|\left|\overrightarrow{BN}\right|\Rightarrow 2(x+1)=2\sqrt{(x-1)^2+y^2}\Rightarrow y^2=4x。$$

(b) 由 $\begin{cases}y^2=4x\\ y=ax+b\end{cases}$ ， 得 $a^2x^2+(2ab-4)x+b^2=0$。⋯⋯(1)

因直線$y=ax+b$ 與 C 相切，(1)有重根，其判別式為 0，

即 $(2ab-4)^2-4a^2b^2=0$ 。因此， $ab=1$。

(c) (i) 解 $\begin{cases}y^2=4x\\ y=mx\end{cases}$ ，得 $x=0$ 或 $x=\frac{4}{m^2}$ 。故除原點外的交點 P 是 $\left(\frac{4}{m^2},\frac{4}{m}\right)$。

(ii) 用（b），設 L_2 為 $y=ax+\frac{1}{a}$。 因 P 是 L_2 上的一點，故 $\frac{4}{m}=a\frac{4}{m^2}+\frac{1}{a}$ ，

並由此得 $4a^2-4am+m^2=0$。 因此， $a=\frac{m}{2}$。

(iii) 因 $m>0$，設 $\tan\theta_1=m$ 及 $\tan\theta_2=\frac{m}{2}$ 分別為 L_1及 L_2 的斜率，

其中 $0<\theta_2<\theta_1<\frac{\pi}{2}$。由

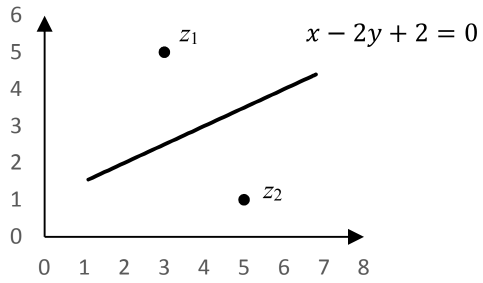
$$\frac{1}{4}=\tan(\theta_1-\theta_2)=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2}=\frac{m-\frac{m}{2}}{1+m(\frac{m}{2})}=\frac{m}{2+m^2}$$

得 $m^2-4m+2=0$。因此， $m=2\pm\sqrt{2}$。

4. (a) (i) 設 $z = x + yi$，則

$$|z - z_1| = |z - z_2|$$
$$\Rightarrow \sqrt{(x-3)^2 + (y-5)^2} = \sqrt{(x-5)^2 + (y-1)^2}$$
$$\Rightarrow (x-3)^2 + (y-5)^2 = (x-5)^2 + (y-1)^2$$
$$\Rightarrow x - 2y + 2 = 0 \text{。}$$

(ii)



(iii) 當 z 為 z_1 和 z_2的中點時，即 $z = 4 + 3i$，$|z - z_1|$ 達到其最小值。
此時，$|z - z_1| = \sqrt{5}$。

(b) (i) $\omega^7 = (\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7})^7 = \cos\frac{14\pi}{7} + i\sin\frac{14\pi}{7} = 1$。

$\omega^7 - 1 = 0 \Rightarrow (\omega - 1)(\omega^6 + \omega^5 + \cdots + 1) = 0$

$\Rightarrow \omega^6 + \omega^5 + \cdots + 1 = 0$ （因 $\omega \neq 1$）。

(ii) $\omega^n + \omega^{-n} = (\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}) + (\cos\frac{2(-n)\pi}{7} + i\sin\frac{2(-n)\pi}{7})$

$= (\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}) + (\cos\frac{2n\pi}{7} - i\sin\frac{2n\pi}{7})$

$= 2\cos\frac{2n\pi}{7}$ 。

(iii) $\left(\cos\frac{2\pi}{7}\right)^2 + \left(\cos\frac{4\pi}{7}\right)^2 + \left(\cos\frac{6\pi}{7}\right)^2$

$= \left(\frac{\omega+\omega^{-1}}{2}\right)^2 + \left(\frac{\omega^2+\omega^{-2}}{2}\right)^2 + \left(\frac{\omega^3+\omega^{-3}}{2}\right)^2$

$= \frac{1}{4}[(\omega^2 + 2 + \omega^{-2}) + (\omega^4 + 2 + \omega^{-4}) + (\omega^6 + 2 + \omega^{-6})]$

$= \frac{1}{4}[(\omega^2 + 2 + \omega^5) + (\omega^4 + 2 + \omega^3) + (\omega^6 + 2 + \omega)]$

$= \frac{1}{4}[5 + (1 + \omega + \omega^2 + \cdots + \omega^6)]$

$= \frac{5}{4}$。

5. (a)

$$\begin{vmatrix} a & b+c & b^2+c^2 \\ b & a+c & a^2+c^2 \\ c & a+b & a^2+b^2 \end{vmatrix} = \begin{vmatrix} a & b+c & b^2+c^2 \\ b-a & a-b & a^2-b^2 \\ c-a & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a & b+c & b^2+c^2 \\ -1 & 1 & a+b \\ -1 & 1 & a+c \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a & b+c & b^2+c^2 \\ -1 & 1 & a^2-b^2 \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a+b+c & b+c & b^2+c^2 \\ 0 & 1 & a^2-b^2 \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c)\text{。}$$

(b) (i) (E) 有唯一解當且僅當 $\begin{vmatrix} 1 & 1 & p \\ p & 1 & 1 \\ 1 & p & 1 \end{vmatrix} \neq 0$，即 $p \neq 1$ 及 $p \neq -2$。

當 $p = 1$，(E) 變成 $\begin{cases} x+y+z=1 \\ x+y+z=1 \\ x+y+z=q \end{cases}$。故 $q = 1$ 及其解為

$x = 1-s-t$， $y = s$，$z = t$， s，$t \in \mathbb{R}$。

(ii) 當 $p = -2$，(E) 變成 $\begin{cases} x+y-2z=1 \\ -2x+y+z=-2 \\ x-2y+z=q \end{cases}$。由這三方程之和得 $0 = q-1$，故 $q = 1$。

然後，解 $\begin{cases} x+y-2z=1 \\ -2x+y+z=-2 \end{cases}$ 得 $x = 1+t$， $y = t$，$z = t$， $t \in \mathbb{R}$。

Suggested Answer:

1. (a) Let M be the mid-point of CE. As ΔCDE is an isosceles triangle with $|CD| = |DE|$, we have $DM \perp CE$. Given $\angle BCA = \frac{\pi}{2}$ and so ΔABC and ΔDBM are similar. Thus, $\frac{|AC|}{|CB|} = \frac{|DM|}{|MB|}$. As $|DM| = \sqrt{|DE|^2 - |ME|^2} = 1$, we get $|AC| = \frac{|DM|}{|MB|}|CB| = \frac{3}{2}$.

   (b) (i) As $|CE|^2 = 4 = |CD|^2 + |DE|^2$, we get $\angle CDE = \frac{\pi}{2}$.

   (ii) From the solution of (a), we have $|DM| = 1$. So, $|DB| = \sqrt{|DM|^2 + |MB|^2} = \sqrt{5}$.
   Hence,
$$\angle CDB = \cos^{-1}\left(\frac{|CD|^2+|BD|^2-|CB|^2}{2|CD||BD|}\right) = \cos^{-1}\left(\frac{2+5-9}{2\sqrt{2}\sqrt{5}}\right) = \cos^{-1}\left(\frac{-1}{\sqrt{10}}\right).$$

   (c) Let X be the point on AB such that $CX \perp AB$. Then, as $PC \perp AB$ (since $PC \perp ABC$), we get $PX \perp AB$. So, the dihedral angle P-AB-C is equal to $\angle PXC$, and $\angle PXC = \tan^{-1}\frac{|PC|}{|CX|}$.
   To find $|CX|$, consider the area of $\angle ABC$. We have $\frac{1}{2}|AC||BC| = \frac{1}{2}|AB||CX|$ and hence $|CX| = \frac{|AC||BC|}{|AB|} = \frac{(\frac{3}{2})3}{\sqrt{\left(\frac{3}{2}\right)^2+3^2}} = \frac{3}{\sqrt{5}}$. Hence, the dihedral angle is $\tan^{-1}\frac{3}{\frac{3}{\sqrt{5}}} = \tan^{-1}\sqrt{5}$.

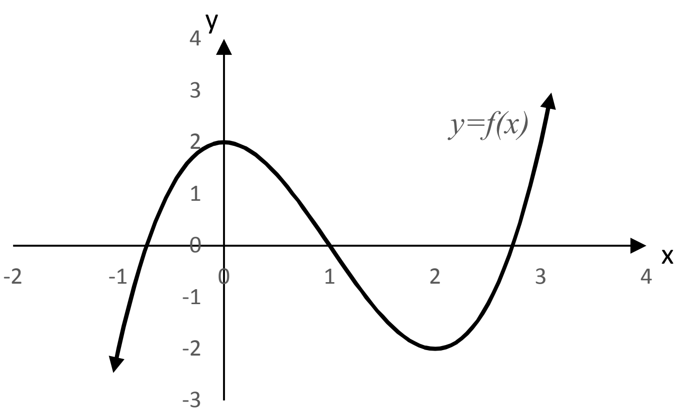
2. (a) (i) As $f(1) = 0$, we know that $x - 1$ is a factor of $f(x)$. By direct calculation, $f(x) = (x-1)(x^2-2x-2)$.
   Hence, $f(x) = 0 \Leftrightarrow x = 1$ or $x^2 - 2x - 2 = 0 \Leftrightarrow x = 1$ or $x = 1 \pm \sqrt{3}$.

   (ii) $f'(x) = 3x^2 - 6x$, $f''(x) = 6x - 6$.

   (iii) $f'(x) = 0 \Leftrightarrow x = 0$ or $x = 2$.
   When $x < 0$, $f'(x) > 0$ and so $f(x)$ is increasing.
   When $0 < x < 2$, $f'(x) < 0$ and so $f(x)$ is decreasing.
   When $2 < x$, $f'(x) > 0$ and so $f(x)$ is increasing.
   Hence, $f(0) = 2$ is a local maximum value, $f(2) = -2$ is a local minimum value.

   (iv) $f''(x) = 0 \Leftrightarrow x = 1$. When $x < 1$, $f''(x) < 0$; when $x > 1$, $f''(x) > 0$.
   Hence, is $(1,0)$ is the inflection point of the curve $y = f(x)$.

   (v)



(b) Solving $\begin{cases} y = x^2 - 8x + 24 \\ y = 8x - x^2 \end{cases}$, we obtain $x = 2$ or $x = 6$.

For $2 < x < 6$, the curve $y = 8x - x^2$ is above the curve $y = x^2 - 8x + 24$.

Hence, the required area is

$$\begin{aligned}\int_2^6 (8x - x^2) - (x^2 - 8x + 24)\,dx &= \int_2^6 -2x^2 + 16x - 24\,dx \\ &= \left[-\tfrac{2}{3}x^3 + 8x^2 - 24x\right]_2^6 \\ &= \tfrac{64}{3}.\end{aligned}$$

3. (a) We have $\overrightarrow{AN} = (x+1, y)$, $\overrightarrow{AB} = (2,0)$ and $\overrightarrow{BN} = (x-1, y)$. Hence,

$$\overrightarrow{AN} \cdot \overrightarrow{AB} = |\overrightarrow{AB}||\overrightarrow{BN}| \Rightarrow 2(x+1) = 2\sqrt{(x-1)^2 + y^2} \Rightarrow y^2 = 4x.$$

(b) From $\begin{cases} y^2 = 4x \\ y = ax + b \end{cases}$, we get $a^2x^2 + (2ab - 4)x + b^2 = 0. \cdots\cdots (1)$

As the line $y = ax + b$ is tangent to *C*, (1) has a double root, its discriminant is 0, i.e., $(2ab-4)^2 - 4a^2b^2 = 0$. Hence, $ab = 1$.

(c) (i) Solving $\begin{cases} y^2 = 4x \\ y = mx \end{cases}$, we get $x = 0$ or $x = \frac{4}{m^2}$. As the intersection point *P* is not the origin, it is $\left(\frac{4}{m^2}, \frac{4}{m}\right)$.

(ii) By (b), let L_2 be $y = ax + \frac{1}{a}$. As *P* is a point on L_2, we get $\frac{4}{m} = a\frac{4}{m^2} + \frac{1}{a}$ which gives $4a^2 - 4am + m^2 = 0$. Hence, $a = \frac{m}{2}$.

(iii) As $m > 0$, let $\tan\theta_1 = m$ and $\tan\theta_2 = \frac{m}{2}$ be the slopes of L_1 and L_2, respectively, where $0 < \theta_2 < \theta_1 < \frac{\pi}{2}$. Then,

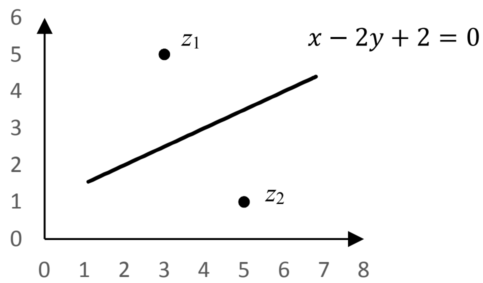
$$\frac{1}{4} = \tan(\theta_1 - \theta_2) = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1\tan\theta_2} = \frac{m - \frac{m}{2}}{1 + m(\frac{m}{2})} = \frac{m}{2 + m^2},$$

from which we get $m^2 - 4m + 2 = 0$. Hence, $m = 2 \pm \sqrt{2}$.

4. (a) (i) Let $z = x + yi$. Then,

$$|z - z_1| = |z - z_2|$$
$$\Rightarrow \sqrt{(x-3)^2 + (y-5)^2} = \sqrt{(x-5)^2 + (y-1)^2}$$
$$\Rightarrow (x-3)^2 + (y-5)^2 = (x-5)^2 + (y-1)^2$$
$$\Rightarrow x - 2y + 2 = 0.$$

(ii)



(iii) $|z - z_1|$ attains it minimum when z is the mid-point of z_1 and z_2, i.e., $z = 4 + 3i$.

In this case, $|z - z_1| = \sqrt{5}$.

(b) (i) $\omega^7 = (\cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7})^7 = \cos\frac{14\pi}{7} + i\sin\frac{14\pi}{7} = 1.$

$\omega^7 - 1 = 0 \Rightarrow (\omega - 1)(\omega^6 + \omega^5 + \cdots + 1) = 0$

$\Rightarrow \omega^6 + \omega^5 + \cdots + 1 = 0$ (since $\omega \neq 1$).

(ii) $\omega^n + \omega^{-n} = (\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}) + (\cos\frac{2(-n)\pi}{7} + i\sin\frac{2(-n)\pi}{7})$

$= (\cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7}) + (\cos\frac{2n\pi}{7} - i\sin\frac{2n\pi}{7})$

$= 2\cos\frac{2n\pi}{7}.$

(iii) $\left(\cos\frac{2\pi}{7}\right)^2 + \left(\cos\frac{4\pi}{7}\right)^2 + \left(\cos\frac{6\pi}{7}\right)^2$

$= \left(\frac{\omega+\omega^{-1}}{2}\right)^2 + \left(\frac{\omega^2+\omega^{-2}}{2}\right)^2 + \left(\frac{\omega^3+\omega^{-3}}{2}\right)^2$

$= \frac{1}{4}[(\omega^2 + 2 + \omega^{-2}) + (\omega^4 + 2 + \omega^{-4}) + (\omega^6 + 2 + \omega^{-6})]$

$= \frac{1}{4}[(\omega^2 + 2 + \omega^5) + (\omega^4 + 2 + \omega^3) + (\omega^6 + 2 + \omega)]$

$= \frac{1}{4}[5 + (1 + \omega + \omega^2 + \cdots + \omega^6)]$

$= \frac{5}{4}.$

5. (a)

$$\begin{vmatrix} a & b+c & b^2+c^2 \\ b & a+c & a^2+c^2 \\ c & a+b & a^2+b^2 \end{vmatrix} = \begin{vmatrix} a & b+c & b^2+c^2 \\ b-a & a-b & a^2-b^2 \\ c-a & a-c & a^2-c^2 \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a & b+c & b^2+c^2 \\ -1 & 1 & a+b \\ -1 & 1 & a+c \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a & b+c & b^2+c^2 \\ -1 & 1 & a^2-b^2 \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (a-b)(a-c)\begin{vmatrix} a+b+c & b+c & b^2+c^2 \\ 0 & 1 & a^2-b^2 \\ 0 & 0 & c-b \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)(a+b+c).$$

(b) (i) (E) has a unique solution if and only if $\begin{vmatrix} 1 & 1 & p \\ p & 1 & 1 \\ 1 & p & 1 \end{vmatrix} \neq 0$, i.e., $p \neq 1$ and $p \neq -2$.

When $p = 1$, (E) becomes $\begin{cases} x+y+z=1 \\ x+y+z=1 \\ x+y+z=q \end{cases}$. So, $q = 1$ and the solution is

$x = 1 - s - t, y = s, z = t,\ \ s, t \in \mathbb{R}$.

(ii) When $p = -2$, (E) becomes $\begin{cases} x+y-2z=1 \\ -2x+y+z=-2 \\ x-2y+z=q \end{cases}$. The sum of all the three equations gives

$0 = q - 1$.

Hence, $q = 1$. Then, solving $\begin{cases} x+y-2z=1 \\ -2x+y+z=-2 \end{cases}$, we get $x = 1 + t, y = t, z = t,\ \ t \in \mathbb{R}$.