澳門大學
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澳門科技大學
UNIVERSIDADE DE CIÊNCIA E TECNOLOGIA DE MACAU
MACAU UNIVERSITY OF SCIENCE AND TECHNOLOGY

澳門四高校聯合入學考試（語言科及數學科）

Joint Admission Examination for Macao Four Higher Education Institutions (Languages and Mathematics)

2020 年試題及參考答案
2020 Examination Paper and Suggested Answer

數學附加卷 Mathematics Supplementary Paper

注意事項：

1. 考生獲發文件如下:

   1.1 本考卷包括封面共 22 版

   1.2 草稿紙一張

2. 請於本考卷封面填寫聯考編號、考場、樓宇、考室及座號。

3. 本考卷共有五條解答題，每題二十分，任擇三題作答。全卷滿分為六十分。

4. 必須在考卷內提供的橫間頁內作答，寫在其他地方的答案將不會獲評分。

5. 必須將解題步驟清楚寫出。只當答案和所有步驟正確而清楚地表示出來，考生方可獲得滿分。

6. 本考卷的圖形並非按比例繪畫。

7. 考試中不可使用任何形式的計算機。

8. 請用藍色或黑色原子筆作答。

9. 考試完畢，考生須交回本考卷及草稿紙。

Instructions:

1. Each candidate is provided with the following documents:

   1.1 Question paper including cover page – 22 pages

   1.2 One sheet of draft paper

2. Fill in your JAE No., campus, building, room and seat no. on the front page of the examination paper.

3. There are 5 questions in this paper, each carries 20 marks. Answer any 3 questions. Full mark of this paper is 60.

4. Put your answers in the lined pages provided. Answers put elsewhere will not be marked.

5. Show all your steps in getting to the answer. Full credits will be given only if the answer and all the steps are correct and clearly shown.

6. The diagrams in this examination paper are not drawn to scale.

7. Calculators of any kind are not allowed in the examination.

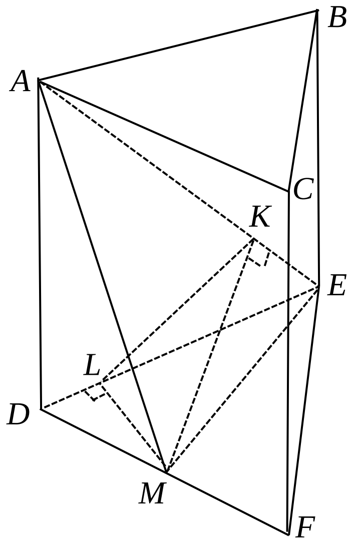
8. Answer the questions with a blue or black ball pen.

9. Candidates must return the question paper and draft paper at the end of the examination.

任擇三題作答，每題二十分。請把答案寫在緊接每條題目之後的 3 頁橫間頁內。

Answer any 3 questions, each carries 20 marks. Write down the answers on the 3 lined pages following each question.

1.



如上圖所示，ABC 是一等邊三角形，其邊長為 2，$ABCDEF$ 是一直三棱柱，其棱長為 3。 M 為 DF 的中點，K 為 AE 上的一點，使得 $MK \perp AE$，L 為 DE 上的一點，使得 $ML \perp DE$。

(a) (i) 證明 $AM \perp EM$ ，從而求三角形 AME 的面積。 (5 分)

(ii) 求三棱錐 A-DEM 的體積，從而求點 D 至平面 AEM 的距離。 (5 分)

(b) 求二面角 $D-AE-M$，答案以 $\sin^{-1}$ 表示。[提示: 證明 $LK \perp AE$。] (10 分)

In the above figure, ABC is an equilateral triangle with edge length 2, $ABCDEF$ is a right triangular prism with edge length 3. M is the mid-point of DF, K is the point on AE such that $MK \perp AE$, and L is the point on DE such that $ML \perp DE$

(a) (i) Show that $AM \perp EM$. Hence find the area of the triangle AME. (5 marks)

(ii) Find the volume of the triangular pyramid A-DEM. Hence find the distance from point D to the plane AEM. (5 marks)

(b) Find the dihedral angle $D-AE-M$. Express your answer in terms of $\sin^{-1}$. [Hint. Show that $LK \perp AE$.] (10 marks)

2. (a) 設 $f(x) = x^4 - 4x^3 + 16$。

(i) 求 $f'(x)$ 及 $f''(x)$。 (2 分)

(ii) 求 $f(x)$ 的局部極大值和局部極小值。 (4 分)

(iii) 求曲線 $y = f(x)$ 的拐點。 (2 分)

(iv) 繪出曲線 $y = f(x)$，$-2 \leq x \leq 4$。 (3 分)

(v) 繪出曲線 $y = f(2x)$，$-1 \leq x \leq 2$。 (1 分)

(b) 求由曲線 $y = x^4 - 4x^3 + 16$ 及直線 $y = 16$ 所包圍的區域的面積。 (8 分)

(a) Let $f(x) = x^4 - 4x^3 + 16$.

(i) Find $f'(x)$ and $f''(x)$. (2 marks)

(ii) Find the local maximum and local minimum values of $f(x)$. (4 marks)

(iii) Find the inflection point(s) of the curve $y = f(x)$. (2 marks)

(iv) Sketch the curve $y = f(x)$, $-2 \leq x \leq 4$. (3 marks)

(v) Sketch the curve $y = f(2x)$, $-1 \leq x \leq 2$. (1 marks)

(b) Find the area of the region bounded by the curve $y = x^4 - 4x^3 + 16$ and the straight line $y = 16$. (8 marks)

3. 已知直線 $L: 3x+y-4=0$ 與圓 $C: x^2+y^2+6x+4y+k=0$ 相交於不同的兩點 $A(x_1, y_1)$ 和 $B(x_2, y_2)$。

(a) 求一條以 x_1 和 x_2 為根的二次方程。 (3 分)

(b) 證明:

(i) $x_1+x_2=3$ 及 $x_1x_2=\frac{32+k}{10}$; (2 分)

(ii) $y_1y_2=\frac{288+9k}{10}-20$; (3 分)

(iii) $x_i^2+y_i^2=6x_i-16-k,\ \ i=1,2$。 (3 分)

(c) 設 O 為原點及 $\angle AOB$ 為一直角。

(i) 求 k 的值。 (4 分)

(ii) 求三角形 AOB 的面積。 (5 分)

Given that the straight line $L: 3x+y-4=0$ and the circle $C: x^2+y^2+6x+4y+k=0$ intersect at two distinct points $A(x_1, y_1)$ and $B(x_2, y_2)$.

(a) Find a quadratic equation with roots x_1 and x_2. (3 marks)

(b) Show that:

(i) $x_1+x_2=3$ and $x_1x_2=\frac{32+k}{10}$; (2 marks)

(ii) $y_1y_2=\frac{288+9k}{10}-20$; (3 marks)

(iii) $x_i^2+y_i^2=6x_i-16-k,\ \ i=1,2$. (3 marks)

(c) Let O be the origin and $\angle AOB$ be a right angle.

(i) Find the value of k. (4 marks)

(ii) Find the area of the triangle AOB. (5 marks)

4. 設複數 $z=\cos\alpha+i\sin\alpha$ ，其中 $i=\sqrt{-1}$。

(a) 用數學歸納法，證明對任意正整數 k，

$$z^k=\cos k\alpha+i\sin k\alpha 。\qquad (6 分)$$

(b) 設 n 為正整數。

(i) 證明 $z\bar{z}=1$ 及 $(1-z)(1-\bar{z})=2-2\cos\alpha$ 。 (3 分)

(ii) 證明若 $z\neq 1$，

$$z+z^2+\cdots+z^n=\frac{z(1-z^n)}{1-z} 。$$

推導出若 $\cos\alpha\neq 1$，

$$\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha=\frac{\cos n\alpha+\cos\alpha-\cos(n+1)\alpha-1}{2-2\cos\alpha} 。\qquad (5 分)$$

(c) 已知恆等式

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \quad 及 \quad \cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2} 。$$

證明若 $\cos\alpha\neq 1$，

$$\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}} 。\qquad (6 分)$$

Let complex number $z=\cos\alpha+i\sin\alpha$, where $i=\sqrt{-1}$.

(a) Using mathematical induction, show that for any positive integer k,

$$z^k=\cos k\alpha+i\sin k\alpha. \qquad (6 marks)$$

(b) Let n be a positive integer.

(i) Show that $z\bar{z}=1$ and $(1-z)(1-\bar{z})=2-2\cos\alpha$. (3 marks)

(ii) Show that if $z\neq 1$,

$$z+z^2+\cdots+z^n=\frac{z(1-z^n)}{1-z}.$$

Deduce that if $\cos\alpha\neq 1$,

$$\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha=\frac{\cos n\alpha+\cos\alpha-\cos(n+1)\alpha-1}{2-2\cos\alpha}. \qquad (5 marks)$$

(iii) Given identities

$$\cos A+\cos B=2\cos\frac{A+B}{2}\cos\frac{A-B}{2} \quad \text{and} \quad \cos A-\cos B=-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}.$$

Show that if $\cos\alpha\neq 1$,

$$\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha=\frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}. \qquad (6 marks)$$

5. (a) 因式分解行列式 $\begin{vmatrix} 1 & 1+a & 1+a^3 \\ 1 & 1+b & 1+b^3 \\ 1 & 1+c & 1+c^3 \end{vmatrix}$。 (8 分)

(b) 設 p 和 q 為常數。已知以 x, y, z 為未知量的方程組 (E): $\begin{cases} x+2y-z=2 \\ x+y+2z=4 \\ px+7y-z=q \end{cases}$。

(i) 求 p 的取值範圍使得 (E) 有唯一解。 (4 分)

(ii) 若 (E) 有多於一個解，求 q 的值，並求 (E) 的通解。 (8 分)

(a) Factorize the determinant $\begin{vmatrix} 1 & 1+a & 1+a^3 \\ 1 & 1+b & 1+b^3 \\ 1 & 1+c & 1+c^3 \end{vmatrix}$. (8 marks)

(b) Let p and q be constants. Given, in unknowns x, y, z, the system of equations

(E): $\begin{cases} x+2y-z=2 \\ x+y+2z=4 \\ px+7y-z=q \end{cases}$.

(i) Find the range of p such that (E) has a unique solution. (4 marks)

(ii) Suppose (E) has more than one solution. Find the value of q, and find the general solution of (E). (8 marks)

參考答案：

1. (a) (i) 從 $EM \perp DF$ 及 $DFE \perp ADFC$ ，得知 $EM \perp ADFC$，故 $EM \perp AM$ 。

$|AM| = \sqrt{|AD|^2 + |DM|^2} = \sqrt{10}$， $|EM| = \sqrt{|DE|^2 - |DM|^2} = \sqrt{3}$。

ΔAME 的面積為 $\frac{1}{2}|AM||EM| = \frac{\sqrt{30}}{2}$ 。

(ii) $A - DEM$ 的體積是 $\frac{1}{3}|AD|(\Delta DEM$ 的面積$) = \frac{\sqrt{3}}{2}$。

設 d 為點 D 至平面 AEM 的距離，則 $D - AEM$ 的體積是 $\frac{d\sqrt{30}}{6}$。

由 $\frac{d\sqrt{30}}{6} = \frac{\sqrt{3}}{2}$，得 $d = \frac{3}{\sqrt{10}}$。

(b) 從 $DEF \perp DEBA$ 及 $ML \perp DE$，得 $ML \perp DEBA$ ，故 LK 為 MK 在平面 $DEBA$ 上的射影。因 $MK \perp AE$，故 $LK \perp AE$。因此，二面角 $D - AE - M = \angle LKM$。

考慮 ΔDEM 的面積，有 $\frac{1}{2}|DE||LM| = \frac{1}{2}|DM||ME|$，從而得出 $|LM| = \frac{\sqrt{3}}{2}$。

考慮 ΔAME 的面積及 (a)(i) 的結果，有 $\frac{1}{2}|AE||MK| = \frac{\sqrt{30}}{2}$，從而得出

$|MK| = \frac{\sqrt{30}}{\sqrt{13}}$。

二面角 $D - AE - M = \angle LKM = \sin^{-1}\frac{|LM|}{|MK|} = \frac{\sqrt{130}}{20}$。

2. (a) (i) $f'(x) = 4x^3 - 12x^2$， $f''(x) = 12x^2 - 24x$。

(ii) $f'(x) = 0 \Leftrightarrow x = 0$ 或 $x = 3$。

當 $x < 0$ 時，$f'(x) < 0$，故 $f(x)$ 是遞減的。

當 $0 < x < 3$ 時，$f'(x) < 0$， 故 $f(x)$ 是遞減的。

當 $3 < x$ 時，$f'(x) > 0$， 故 $f(x)$ 是遞增的。

因此，$f(3) = -11$ 是一局部極小值。

$f''(x) = 0 \Leftrightarrow x = 0$ 或 $x = 2$。

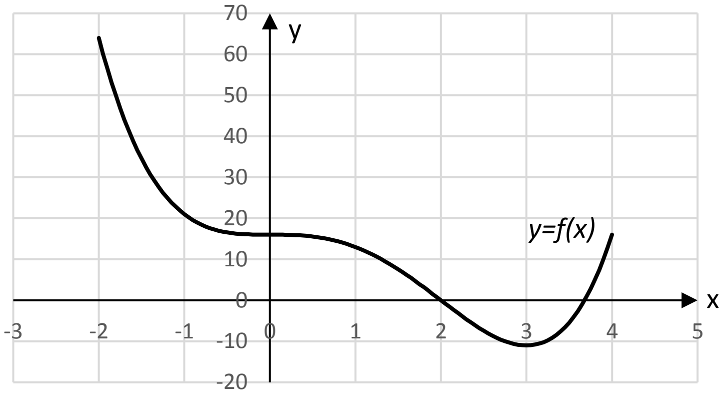
當 $x < 0$ 時，$f''(x) > 0$，故曲線 $y = f(x)$ 是凸的。

當 $0 < x < 2$ 時，$f''(x) < 0$，故 曲線 $y = f(x)$ 是凹的。

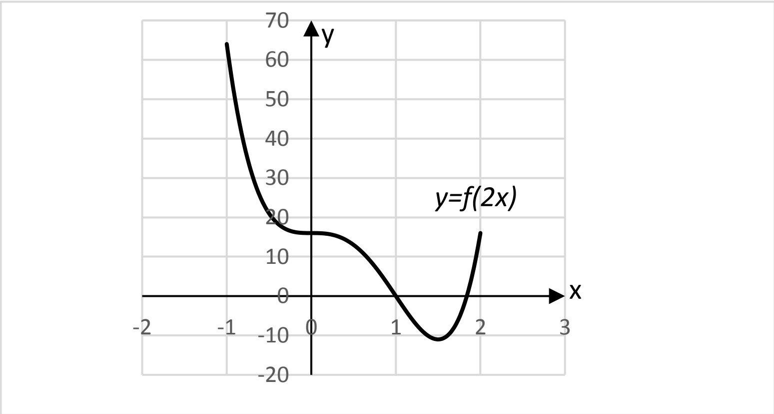
當 $2 < x$ 時，$f''(x) > 0$，故曲線 $y = f(x)$ 是凸的。

因此，曲線 $y = f(x)$ 的拐點是 $(0,16)$ 和 $(2,0)$。

(iii)



(iv)



(b) 解 $\begin{cases} y = x^4 - 4x^3 + 16 \\ y = 16 \end{cases}$，得 $x = 0$ 或 $x = 4$。

當 $0 < x < 4$，曲線 $y = x^4 - 4x^3 + 16$ 是在直線 $y = 16$ 之下，故所求面積為

$$\int_0^4 16 - (x^4 - 4x^3 + 16)dx$$

$$= \int_0^4 4x^3 - x^4 dx$$

$$= \left[x^4 - \frac{x^5}{5}\right]_0^4$$

$$= 51\frac{1}{5}$$

3. (a) (i) 由 $\begin{cases} 3x + y - 4 = 0 \\ x^2 + y^2 + 6x + 4y + k = 0 \end{cases}$，得 $10x^2 - 30x + 32 + k = 0$。

(b) (i) $x_1 + x_2 = \frac{-(-30)}{10} = 3$，$x_1 x_2 = \frac{32+k}{10}$。

(ii) $y_1 y_2 = (4 - 3x_1)(4 - 3x_2) = 16 - 12(x_1 + x_2) + 9x_1 x_2 = \frac{288+9k}{10} - 20$ 。

(iii) 對 $i = 1, 2$，

$$x_i^2 + y_i^2 = -6x_i - 4y_i - k = -6x_i - 4(4 - 3x_i) - k = 6x_i - 16 - k\text{。}$$

(c) (i) 若 *OA* 和 *OB* 其中一條是水平線，則另一條是垂直線。在這情況下，

$x_1x_2 = 0$ 和 $y_1y_2 = 0$。從 (b) (i) 及 (ii) 可得出矛盾。

設 *OA* 和 *OB* 的斜率分別為 m_1 及 m_2。則有

$$m_1m_2 = -1 \Rightarrow \frac{y_1y_2}{x_1x_2} = -1 \Rightarrow \frac{288+9k}{10} - 20 = -\frac{32+k}{10} \Rightarrow k = -12。$$

(ii) 當 $k = -12$，$x_1x_2 = 2$。ΔOAB 的面積是

$$\frac{1}{2}|OA||OB| = \frac{1}{2}\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = \frac{1}{2}\sqrt{6x_1 - 4}\sqrt{6x_2 - 4}$$

$$= \sqrt{9x_1x_2 - 6(x_1 + x_1) + 4} = \sqrt{9(2) - 6(3) + 4} = 2。$$

4. (a) 設 $P(k)$ 代表命題"$z^k = \cos k\alpha + i\sin k\alpha$"。當 $k = 1$，*P*(1) 明顯成立。

假設對某正整數k，$P(k)$ 成立，即 $z^k = \cos k\alpha + i\sin k\alpha$。則

$$z^{k+1} = z^kz$$

$$= (\cos k\alpha + i\sin k\alpha)(\cos\alpha + i\sin\alpha)$$

$$= (\cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha) + i(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha)$$

$$= \cos(k+1)\alpha + i\sin(k+1)\alpha。$$

故 $P(k+1)$ 也成立。

根據數學歸納法原理，$P(k)$ 對任意正整數 k 都成立。

(b)(i) $z\bar{z} = (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha) = \cos^2\alpha + \sin^2\alpha = 1$。

$(1-z)(1-\bar{z}) = 1 - (z+\bar{z}) + z\bar{z} = 1 - 2\cos\alpha + 1 = 2 - 2\cos\alpha$。

(ii) 因 $z \neq 1$，從

$$(1-z)(z + z^2 + \cdots + z^n) = z - z^{n+1} = z(1-z^n)$$

中可得出結果。

因 $\cos\alpha \neq 1$，即 $z \neq 1$，故

$$\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \mathrm{Re}(z + z^2 + \cdots + z^n)$$

$$= \mathrm{Re}\frac{z(1-z^n)}{1-z} = \mathrm{Re}\frac{z(1-z^n)(1-\bar{z})}{(1-z)(1-\bar{z})} = \frac{\mathrm{Re}(z^n + z - z^{n+1} - 1)}{2 - 2\cos\alpha} = \frac{\cos n\alpha + \cos\alpha - \cos(n+1)\alpha - 1}{2 - 2\cos\alpha}。$$

(iii)

$$\begin{aligned}\cos\alpha+\cos 2\alpha+\cdots+\cos n\alpha &= \frac{\cos n\alpha+\cos\alpha-\cos(n+1)\alpha-1}{2-2\cos\alpha}\\ &= \frac{2\cos\frac{(n+1)\alpha}{2}\cos\frac{(n-1)\alpha}{2}-2\cos^2\frac{(n+1)\alpha}{2}}{4\sin^2\frac{\alpha}{2}}\\ &= \frac{2\cos\frac{(n+1)\alpha}{2}\left[\cos\frac{(n-1)\alpha}{2}-\cos\frac{(n+1)\alpha}{2}\right]}{4\sin^2\frac{\alpha}{2}}\\ &= \frac{2\cos\frac{(n+1)\alpha}{2}\left[-2\sin\frac{n\alpha}{2}\sin\frac{(-\alpha)}{2}\right]}{4\sin^2\frac{\alpha}{2}}\\ &= \frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}\ 。\end{aligned}$$

5. (a)

$$\begin{aligned}\begin{vmatrix}1 & 1+a & 1+a^3\\ 1 & 1+b & 1+b^3\\ 1 & 1+c & 1+c^3\end{vmatrix} &= \begin{vmatrix}1 & 1+a & 1+a^3\\ 0 & b-a & b^3-a^3\\ 0 & c-a & c^3-a^3\end{vmatrix}\\ &= \begin{vmatrix}1 & 1+a & 1+a^3\\ 0 & b-a & (b-a)(b^2+ba+a^2)\\ 0 & c-a & (c-a)(c^2+ca+a^2)\end{vmatrix}\\ &= (b-a)(c-a)\begin{vmatrix}1 & 1+a & 1+a^3\\ 0 & 1 & b^2+ba+a^2\\ 0 & 1 & c^2+ca+a^2\end{vmatrix}\\ &= (b-a)(c-a)(c^2+ca-b^2-ba)\\ &= (b-a)(c-a)[(c+b)(c-b)+a(c-b)]\\ &= (a-b)(b-c)(c-a)(a+b+c)\ 。\end{aligned}$$

(b) (i) (E) 有唯一解的條件是 $\begin{vmatrix}1 & 2 & -1\\ 1 & 1 & 2\\ p & 7 & -1\end{vmatrix}\neq 0$，即 $5p-20\neq 0$。

p 的取值範圍是 $\{p: p\neq 4\}$。

(ii) 設 $p=4$，從 (E) 得 $\begin{cases}x+2y-z & = & 2\\ -y+3z & = & 2\\ -y+3z & = & q-8\end{cases}$ 。

從以上的第二及第三條方程得 $q=10$。

解 $\begin{cases}x+2y-z=2\\ -y+3z=2\end{cases}$，得 $x=6-5t$，$y=3t-2$，$z=t$，其中 t 為任意實數。

Suggested Answer

1. (a) (i) From $EM \perp DF$ and $DFE \perp ADFC$, we have $EM \perp ADFC$ and hence $EM \perp AM$.

$|AM| = \sqrt{|AD|^2 + |DM|^2} = \sqrt{10}$, $|EM| = \sqrt{|DE|^2 - |DM|^2} = \sqrt{3}$.

The area of ΔAME is $\frac{1}{2}|AM||EM| = \frac{\sqrt{30}}{2}$.

(ii) The volume of $A - DEM$ is $\frac{1}{3}|AD|(\text{area of } \Delta DEM) = \frac{\sqrt{3}}{2}$.

Let d be the distance from point D to plane AEM. Then, the volume of $D - AEM$ is $\frac{d\sqrt{30}}{6}$. From $\frac{d\sqrt{30}}{6} = \frac{\sqrt{3}}{2}$, we get $d = \frac{3}{\sqrt{10}}$.

(b) From $DEF \perp DEBA$ and $ML \perp DE$, we get $ML \perp DEBA$ and hence LK is the orthogonal projection of MK in the plane $DEBA$. Since $MK \perp AE$, $LK \perp AE$.

Hence the dihedral angle $D - AE - M = \angle LKM$.

Considering the area of ΔDEM, we have $\frac{1}{2}|DE||LM| = \frac{1}{2}|DM||ME|$ and hence $|LM| = \frac{\sqrt{3}}{2}$.

Considering the area of ΔAME and the result in (a)(i), we have $\frac{1}{2}|AE||MK| = \frac{\sqrt{30}}{2}$ and hence $|MK| = \frac{\sqrt{30}}{\sqrt{13}}$.

The dihedral angle $D - AE - M = \angle LKM = \sin^{-1}\frac{|LM|}{|MK|} = \frac{\sqrt{130}}{20}$.

2. (a) (i) $f'(x) = 4x^3 - 12x^2$, $f''(x) = 12x^2 - 24x$.

(ii) $f'(x) = 0 \Leftrightarrow x = 0$ or $x = 3$.

When $x < 0$, $f'(x) < 0$ and hence $f(x)$ is decreasing.

When $0 < x < 3$, $f'(x) < 0$ and hence $f(x)$ is decreasing.

When $3 < x$, $f'(x) > 0$ and hence $f(x)$ is increasing.

Hence $f(3) = -11$ is a local minimum value.

$f''(x) = 0 \Leftrightarrow x = 0$ or $x = 2$.

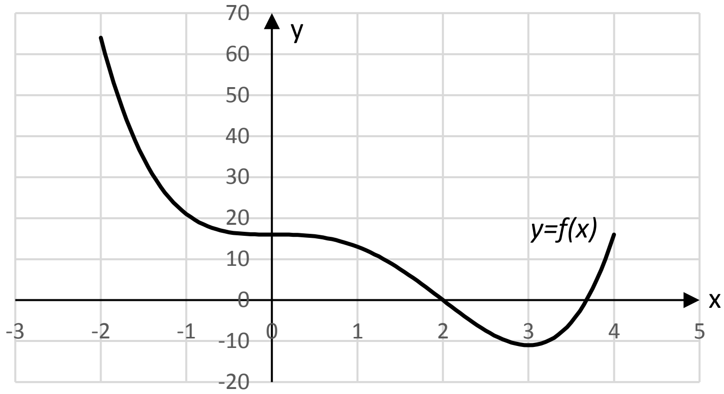
When $x < 0$, $f''(x) > 0$ and hence the curve $y = f(x)$ is convex.

When $0 < x < 2$, $f''(x) < 0$ and hence the curve $y = f(x)$ is concave.

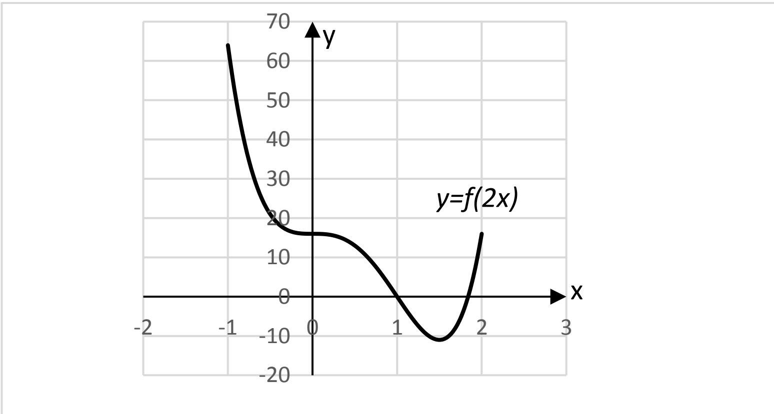
When $2 < x$, $f''(x) > 0$ and hence the curve $y = f(x)$ is convex.

Hence, the inflection points of the curve $y = f(x)$ are $(0,16)$ and $(2,0)$.

(iii)



(iv)



(b) Solving $\begin{cases} y = x^4 - 4x^3 + 16 \\ y = 16 \end{cases}$, we get $x = 0$ or $x = 4$.

When $0 < x < 4$, the curve $y = x^4 - 4x^3 + 16$ is below the line $y = 16$.

Hence, the required area is

$$\int_0^4 16 - (x^4 - 4x^3 + 16)dx$$

$$= \int_0^4 4x^3 - x^4 dx$$

$$= \left[x^4 - \frac{x^5}{5}\right]_0^4$$

$$= 51\frac{1}{5}$$

3. (a) (i) From $\begin{cases} 3x + y - 4 = 0 \\ x^2 + y^2 + 6x + 4y + k = 0 \end{cases}$, we get $10x^2 - 30x + 32 + k = 0$.

(b) (i) $x_1 + x_2 = \frac{-(-30)}{10} = 3,\ x_1x_2 = \frac{32+k}{10}$.

(ii) $y_1y_2 = (4 - 3x_1)(4 - 3x_2) = 16 - 12(x_1 + x_2) + 9x_1x_2 = \frac{288+9k}{10} - 20$.

(iii) For $i = 1, 2$,

$$x_i^2 + y_i^2 = -6x_i - 4y_i - k = -6x_i - 4(4 - 3x_i) - k = 6x_i - 16 - k.$$

(c) (i) If one of *OA* and *OB* is horizontal, then the other one is vertical. In this case, $x_1x_2 = 0$ and $y_1y_2 = 0$. Form (b) (i) and (ii), we get a contradiction.

Suppose the slopes of *OA* and *OB* are m_1 and m_2, respectively. We have

$$m_1m_2 = -1 \Rightarrow \frac{y_1y_2}{x_1x_2} = -1 \Rightarrow \frac{288+9k}{10} - 20 = -\frac{32+k}{10} \Rightarrow k = -12.$$

(ii) When $k = -12,\ x_1x_2 = 2$. The area of ΔOAB is

$$\frac{1}{2}|OA||OB| = \frac{1}{2}\sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2} = \frac{1}{2}\sqrt{6x_1 - 4}\sqrt{6x_2 - 4}$$

$$= \sqrt{9x_1x_2 - 6(x_1 + x_1) + 4} = \sqrt{9(2) - 6(3) + 4} = 2.$$

4. (a) Let $P(k)$ be the proposition " $z^k = \cos k\alpha + i\sin k\alpha$ ". When $k = 1,\ P(1)$ is obviously true. Suppose $P(k)$ is true for some positive integer k. That is, $z^k = \cos k\alpha + i\sin k\alpha$. Then,

$$\begin{aligned} z^{k+1} &= z^k z \\ &= (\cos k\alpha + i\sin k\alpha)(\cos\alpha + i\sin\alpha) \\ &= (\cos k\alpha\cos\alpha - \sin k\alpha\sin\alpha) + i(\sin k\alpha\cos\alpha + \cos k\alpha\sin\alpha) \\ &= \cos(k+1)\alpha + i\sin(k+1)\alpha. \end{aligned}$$

Hence $P(k+1)$ is also true.

By the principle of mathematical induction, $P(k)$ is true for any positive integer k.

(b)(i) $z\bar{z} = (\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha) = \cos^2\alpha + \sin^2\alpha = 1$.

$$(1 - z)(1 - \bar{z}) = 1 - (z + \bar{z}) + z\bar{z} = 1 - 2\cos\alpha + 1 = 2 - 2\cos\alpha.$$

(ii) As $z \neq 1$, the result follows from

$$(1 - z)(z + z^2 + \cdots + z^n) = z - z^{n+1} = z(1 - z^n).$$

As $\cos\alpha \neq 1$, which means $z \neq 1$, we have

$$\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \mathrm{Re}(z + z^2 + \cdots + z^n)$$

$$= \mathrm{Re}\frac{z(1-z^n)}{1-z} = \mathrm{Re}\frac{z(1-z^n)(1-\bar{z})}{(1-z)(1-\bar{z})} = \frac{\mathrm{Re}(z^n + z - z^{n+1} - 1)}{2 - 2\cos\alpha} = \frac{\cos n\alpha + \cos\alpha - \cos(n+1)\alpha - 1}{2 - 2\cos\alpha}.$$

(iii)

$$\cos\alpha + \cos 2\alpha + \cdots + \cos n\alpha = \frac{\cos n\alpha + \cos\alpha - \cos(n+1)\alpha - 1}{2 - 2\cos\alpha}$$

$$= \frac{2\cos\frac{(n+1)\alpha}{2}\cos\frac{(n-1)\alpha}{2} - 2\cos^2\frac{(n+1)\alpha}{2}}{4\sin^2\frac{\alpha}{2}}$$

$$= \frac{2\cos\frac{(n+1)\alpha}{2}\left[\cos\frac{(n-1)\alpha}{2} - \cos\frac{(n+1)\alpha}{2}\right]}{4\sin^2\frac{\alpha}{2}}$$

$$= \frac{2\cos\frac{(n+1)\alpha}{2}\left[-2\sin\frac{n\alpha}{2}\sin\frac{(-\alpha)}{2}\right]}{4\sin^2\frac{\alpha}{2}}$$

$$= \frac{\sin\frac{n\alpha}{2}\cos\frac{(n+1)\alpha}{2}}{\sin\frac{\alpha}{2}}.$$

5. (a)

$$\begin{vmatrix} 1 & 1+a & 1+a^3 \\ 1 & 1+b & 1+b^3 \\ 1 & 1+c & 1+c^3 \end{vmatrix} = \begin{vmatrix} 1 & 1+a & 1+a^3 \\ 0 & b-a & b^3-a^3 \\ 0 & c-a & c^3-a^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1+a & 1+a^3 \\ 0 & b-a & (b-a)(b^2+ba+a^2) \\ 0 & c-a & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} 1 & 1+a & 1+a^3 \\ 0 & 1 & b^2+ba+a^2 \\ 0 & 1 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)(c^2+ca-b^2-ba)$$

$$= (b-a)(c-a)[(c+b)(c-b)+a(c-b)]$$

$$= (a-b)(b-c)(c-a)(a+b+c).$$

(b) The condition that (E) has a unique solution is $\begin{vmatrix} 1 & 2 & -1 \\ 1 & 1 & 2 \\ p & 7 & -1 \end{vmatrix} \neq 0$, i.e. $5p - 20 \neq 0$.

The range of p is $\{p : p \neq 4\}$.

(ii) Let $p = 4$. From (E) we obtain $\begin{cases} x + 2y - z = 2 \\ \quad -y + 3z = 2 \\ \quad -y + 3z = q - 8 \end{cases}$.

From the second and third equations above, we get $q = 10$.

Solving $\begin{cases} x + 2y - z = 2 \\ \quad -y + 3z = 2 \end{cases}$, we get $x = 6 - 5t$, $y = 3t - 2$, $z = t$, where t is any real number.